V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

MATEMATIKA INSTITUTI

RO‘ZIBOYEV MARKS BOTIRBOEVICH

DINAMIK SISTEMALARDA TURG‘UNLIK, ERGODIKLIK VA BOSHQARUV

01.01.02 – Differensial tenglamalar va matematik fizika

FIZIKA-MATEMATIKA FANLARI DOKTORI (DSc) DISSERTATSIYASI AVTOREFERATI

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Fizika-matematika fanlari doktori (DSc) dissertatsiyasi mavzusi O'zbekiston Respublikasi oliy ta'lim fan va innovatsiyalar vazirligi huzurigadi Oliy attestatsiya komissiyasida B2024.3DSc/FM274 raqam bilan ro'yxatga olingan.

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Ilmiy maslahatchi: **Azamov Abdulla Azamovich**
fizika-matematika fanlari doktori, akademik

Rasmiy opponentlar: **Djalilov Akhtam Abdurakhmanovich**
fizika-matematika fanlari doktori, professor

Durdiyev Durdimurad Kalandarovich
fizika-matematika fanlari doktori, professor

Sachkov Yuriy Leonidovich
fizika-matematika fanlari doktori, professor

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Doktorlik dissertatsiyasi bilan V.I. Romanovskiy nomidagi Matematika institutining Axborot-resurs markazida tanishish mumkin (192-raqami bilan ro'yxatga olingan). (Manzil: 100174, Toshkent sh., Olmazor tumani, Universitet ko'chasi, 9-uy. Tel.: (+99871) 207-91-40).

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O'ZBEKISTON RESPUBLIKASI FANLAR AKADEMIYASI V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI

U. A. Roziqov
Ilmiy darajalar beruvchi Ilmiy kengash raisi, f.-m.f.d., akademik

J. K. Adashev
Ilmiy darajalar beruvchi Ilmiy kengash ilmiy kotibi,f.-m.f.d., katta ilmiy xodim

R. R. Ashurov
Ilmiy darajalar beruvchi Ilmiy kengash huzuridagi Ilmiy seminar raisi o'rinbosari, f.-m.f.d. professor

KIRISH (doktorlik dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zarurati. Jahon miqyosida olib borilayotgan ilmiy-amaliy tadqiqotlarning salmoqli qismi evolyutsion xarakterga ega boʻlib, vaqt oʻtishi bilan muayyan qonuniyat asosida oʻzgaradi. Bunday oʻzgarish dinamik jarayon deyiladi. Dinamik jarayonlar differensial tenglamalar bilan modellashtiriladi. Agar jarayonlar qanday kechishini oʻrganish maqsad qilinsa, bunga eng samarali vositalardan biri dinamik sistemalar nazariyasidir. Dinamik jarayonlarni matematik modellashtirishning mohiyati sistemaning qonuniyatlarini topish, sistemani xarakterlovchi parametrlarni tahlil qilishdan iborat. Tabiiy jarayonlarni matematik modellashtirishda parameterlarning qiymatlarini taqriban hisoblashga toʻgʻri keladi. Bunda parameterlar qiymatlarining kichik oʻzgarishlari taʻsirida matematik modellar dinamikasining qanday oʻzgarishini tadqiq qilish masalasi paydo boʻladi. Bu masala dinamik sistemalarning turgʻunlik masalasi deyiladi va u muhim amaliy ahamiyat kasb etadi. Jumladan dinamik sistema evolyutsiyasini tavsiflash va uning tashqi taʻsirlar ostida turgʻunligini tadqiq qilish hozirgi kunning dolzarb muammolaridan hisoblanadi. Bunda “turgʻunlik“ tushunchasini turli yoʻsinda aniqlashtirish mumkin va shu tarzda dinamik sistemalar nazariyasining u yoki bu yoʻnalishi hosil boʻladi.

Hozirgi kunda jahonda dinamik sistemalar nazariyasi va ularning amaliy masalalarga tatbiqlari keng oʻrganilmoqda. Dinamik sistemalar nazariyasiga 19-asr oxirlarida A. Puankare tomonidan asos solingan. Shu nuqtai nazardan u matematikaning klassik sohasiga aylangan. Shunga qaramay, masalalarning xilma-xilligi va koʻplab sohalarga tatbiqlari tufayli u hozirgi zamon matematikasining jadal rivojlanayotgan eng dolzarb sohalaridan biriligicha qolmoqda. Oxirgi oʻttiz yilda matematika yoʻnalishidagi muhim ilmiy natijalar uchun ta’sis etilgan Fields mukofoti sovindorlaridan yettitasi dinamik sistemalar nazariyasi boʻyicha mutaxasislar ekanligi shundan dalolat beradi.

Mamlakatimizda oxirgi yillarda fundamental tadqiqotlarni jahonda yetakchi oʻrinlarga chiqlarishga katta eʻtibor berilayotganligi[1] tufayli dinamik va boshqariluvchi sistemalarning turgʻunlik nazariyasi boʻyicha salmoqli natijalarga erishildi. Xalqaro matematika jamiyati e'tiborida turgan sohalardan biri boʻlgan dinamik sistemalar nazariyasi boʻyicha tatqiqotlarni yurtimizda yanada kuchaytirish, jumladan dinamik sistemalarning parametrlar boʻyicha turgʻunligini tadqiq qilish muhim ahamiyatga ega. Mazkur dissertatsiya ishida olib borilgan ilmiy tadqiqotlar ana shu masalalar turkumiga mansub natijalardan hisoblnaladi.

Oʻzbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi “Oʻzbekiston Respublikasini yanada rivojlantirish boʻyicha harakatlar strategiyasi toʻgʻrisida”gi PF-4947-son Farmoni, 2019-yil 9-iyuldagi “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qoʻllab-quvvatlash, shuningdek, Oʻzbekiston Respublikasi Fanlar Akademiyasining V.I. Romanovskiy nomidagi Matematika

[1] Oʻzbekiston Respublikasi Prezidentining 2019-yil 9-iyuldagi “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qoʻllab-quvvatlash shuningdek, Oʻzbekiston Respublikasi Fanlar akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari toʻgʻrisida” gi № PQ-4387-son qarori

instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida”gi PQ-4387-son Qarori va 2020-yil 7-maydagi “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi PQ-4708-sonli Qarori hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi. Mazkur tadqiqot respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Dissertatsiya mavzusi bo‘yicha xorijiy ilmiy tadqiqotlar sharxi. Dinamik sistemalar nazariyasi bo‘yicha ilmiy izlanishlar dunyoning yetakchi ilmiy markazlarida, jumladan, AQShning yetakchi universitetlari: Princeton, Masachuest Technologiya Universiteti, Maryland, Pennsilvaniya davlat universitetlari, Georgia texnika instituti, Fransiyaning Sorbonna universiteti, Shveytsariyaning Zurich universiteti, Buyuk Britaniyaning Warvick universiteti, Rossiya Fanlar akademiyasining V.A. Setklov nomidagi Matematika instituti, Yaponiyaning Kyushu, Keio universitetlari, Xitoyning Southern University of Science and Technology universiteti kabi ilg‘or ilmiy markazlarida sohaning turli yo‘nalishlari bo‘yicha keng miqyosda ilmiy tadqiqotlar olib borilmoqda.

So‘nggi yillarda dinamik sistemalarni statistik nuqtai nazardan o‘rganish bo‘yicha bir qator muhim ilmiy natijalar olindi. Jumladan, geometrik Lorents attractori uchun korrelyatsiya funksiyasining kamayish tezligi exponensial ekanligi isbotlandi (Warwick universiteti, Buyuk Britaniya), “Sinai billiardi” deb ataladigan sistema oqimi uchun korrelyatsiya funksiyasining eksponensial tezlikda kamayishi isbotlandi (Italiyadagi Tor-Vergata, Fransiyadagi Sorbonna, va AQShdagi New York univesiteti olimlari hamkorlikda). Bu natijalar dinamik sistemalar vaqt o‘tishi bilan muvozanat o‘lchoviga eksponensial yaqinlashishini anglatadi. Bundan tashqari silliq Anosov diffeomorfizmlari uchun invariant o‘lchov tashqi ta'sirlarga nisbatan turg‘un va hatto silliq ekanligi isbotlandi (Institut des Hautes Etudes Scientifiques, Fransiya).

Shuni ta‘kidlash lozimki so‘nggi yillarda stoxastik differensial tenglamalarning egizi bo‘lmish tasodifiy dinamik sistemalar nazariyasi ham jadal rivonlanmoqda. Bu yo‘nalish bo‘yicha Australia, Italiya, Fransiya, Xorvatiya, Buyuk Britaniya va AQShning yetakchi ilmiy markazlarida ilmiy izlanishlar olib borilmoqda.

Muammoning o‘rganilganlik darajasi. Dinamik sistemalarda turg‘unlik nazariyasi A. Puankare, A.M.Lyapunov, A.Andronov, L.S. Pontryaginlar nomi bilan bog‘liq. Lyapunov ma‘nosida turg‘unlik deganda biz sistema yechiming topologik turg‘unligini tushunamiz. Turg‘unlik harakat barqarorligining matematik ifodasi bo‘lganligi sababli mexanikaning ko‘plab masalalarida o‘z tadbiqini topdi. Keyinchalik, 1937 yilda Pontryagin va Andronovlar tomonidan dinamik sistema trayektoriyalari tuzilishining turg‘unligi tushunchasi kiritilib, tekislikda sistemalar tuzilishining turg‘unligi uchun zaruriy va yetarli shartni topishga erishildi. O‘tgan asrning 60-yillarida M. Peixoto va M.Ch. Peixotolar tomonidan dinaimik sistema tuzilishining turg‘un bo‘lishi uchun yetarli shartlar topildi va bu shartlarni qonatlantiruvchi sistemalar diffeomorfizmlar oilasida hamma yerda zich ekanligini ko‘rsatildi. Shu yillari D.Anosov torning giperbolik avtomorfizmlari tuzilishi tug‘run ekanini isbotladi va bu sinfni umulashtirdi, so‘ngra bu sinf Anosov diffeomorfizmlari deb atala boshlandi. S.Smale bu tushunchani yanada umulashtirib, “Aksioma–A” deb nomlangan sinfni kiritdi. S.Smale va uning ilmiy maktabi vakillari tomonidan yuqori o‘lchovli fazolarda Andoronov-Pontryagin sistemalarining umulashmasi bugungi kunda Morse-Smale sistemalari deb ataluvchi sinf ekanini ko‘rsatildi.

Dinamik sistemalarning topologik xossalari bilan bir vaqtta giperbolik dinamik sistemalarni statistik nuqtai nazardan tadqiq qilina boshlandi. E.Hopf, D. Anosovlar giperbolik sistemalarning ergodik ekanini ko‘rsatgan bo‘lsa, Ya. Sinai tomonidan torning avtomorfizmlari uchun Markov bo‘laklashlarining kiritilishi dinamikani tadqiq qilishga statistik fizika usullarini qo‘llash imkonini berdi. Keyinchalik Markov bo‘laklashlari R.Bowen tomonidan “Aksioma–A” sistemalari uchun qurildi. Ular uchun maxsus Sinai-Ruelle-Bowen o‘lchovlari mavjud bo‘lishi va korrelatsiya funksiyasining exponentsial kamayishi, markaziy limit teorema va yana bir qancha statistik xossalar isbotlandi. Giperbolik sistemalarning geometriya va spectral nazariya bilan bog‘likligi sohaga ko‘plab yangi go‘yalar olib keldi. Masalan, Gauss egriligi manfiy bo‘lgan ko‘pxillik ustidagi geodezik chiqizlar Anosov (giperbolik) oqimi bo‘ladi va bu oqim ergodik, uning korrelatsiya koeffitsientlari exponensial kamayadi hamda ko‘pxillikda aniqlangan Laplas-Beltrami operatorining xos sonlari oqim aniqlagan transfer operatorining xos sonlariga bog‘liq bo‘ladi. Giperbolik sistemalarning eng katta kamchiligi, ular xar qanday ko‘pxillik ustida ham aniqlanavermaydi. Shuning uchun oxirgi 30 yilda giperbolik sistemalarni o‘z ichiga olgan, holatlar fazosi ixtiyoriy ko‘pxillik bo‘la oladigan kattaroq sinfni aniqlash ustida ilmiy tadqiqot ishlari qizg‘in davom etmoqda.

Hozirgi kunda sohaning turli yo‘nalishlari bo‘yicha yurtimizda va horijda ko‘plab ilmiy maktablar faoliyat yuritmodqa. Turg‘unlik nazariyasi va uning bevosita davomi bo‘lmish optimal boshqaruv nazariyasi va differensial o‘yinlarga horijlik hamkasblar bilan birgalikda yurtimiz olimlari katta hissa qo‘shdilar. N.Satimov tomonidan quvishning 3-usuli deb ataluvchi usul yaratildi, bundan tashqari A.Azamov tomonidan tug‘unlik nazariyasi bo‘yicha umumlashgan xarakteristik ko‘rsatkichlarning turg‘unlik nazariyasiga tadbiqlari bo‘yicha, keyinroq esa Pontryagin taklif etgan alternirlangan integralning xossalari haqida muhim natijalar olindi. Keyinchalik differensial o‘yinlar nazariyasi rivojlanib, cheksiz o‘lvochli sistemalar bilan tavsiflanuvchi sistemalar uchun, Riman ko‘pxilliklarida berilgan sistemalar va inertsiyali ob‘ektlar harakitini ifodalovchi sistemalar hamda kechikuvchi parametrli sistemalar uchun differensial o‘yin masalalari bo‘yicha M.To‘xtasinov, G.Ibragimov, O.Qo‘chqorov, B.Samatov va N.Mamadaliyevlar tomonidan salmoqli natijalar olindi.

Dissertatsiya tadqiqotining dissertatsiya bajarilgan ilmiy-tadqiqot muassasasining ilmiy tadqiqot ishlari rejalari bilan bog‘liqligi. Dissertatsiya tadqiqoti V.I.Romanovskiy nomidagi Matematika instituti ilmiy tadqiqot rejasidagi

OT-Ф4-84 “Polinomial sistemalar uchun diskret-sonli usul hamda uning siklik va boshqariluvchi jarayonlarni modellashtirishga tatbiqlari” (2017-2020 yillar) mavzusidagi fundamental loyihasi, “Dinamik jarayonlarning matematik modellari va ularning tatbiqlari” (2020-2024 yillar) mavzusidagi tadqiqot dasturlari boʻyicha olib borilgan ilmiy tadqiqot ishlari bilan bevosita bogʻliq.

Tadqiqotning maqsadi statistik turgʻunlik, invariant oʻlchovning tashqi parametr boʻyicha uzluksizligi va silliqligi, invariant oʻlchovlarning asimptotik turgʻunligi va boshqariluvchanlik masalalarini yechishdan iborat.

Tadqiqotning vazifalalri:

cheksiz oʻlchovli fazolarda aniqlangan boshqariluvchan sistemalarda asimptotik turgʻinlik masalalarini tadqiq qilish;

chekli oʻlchovli fazoda aniqlangan, oʻyinchilarning boshqaruviga integral cheklovlar qoʻyilgan differensial oʻyinlarda quvish-qochish masalasini yechish;

Lorenz attraktorining statistik turgʻunligini isbotlash;

tasodifiy dinamik sistemalarda statsionar oʻlchov silliqligini isbotlash uchun abstrakt usul ishlab chiqish;

tasodifiy dinamik sistemalar uchun deyarli barcha realizatsizyalarda korrelyatsiya funksiyasining kamayish tezligini baholash uchun abstrakt Yong minorasining tasodifiy analogini ishlab chiqish va uni kuchsiz giperbolik boʻlgan sistemalarda qoʻllashdan iborat.

Tadqiqotning obʻekti. Chiziqli boshqaruv sistemalari va kuchsiz giperbolik xaotik dinamik sistemalar.

Tadqiqotning predmeti. Dinamik va boshqariluvchi sistemalarning turgʻunligi.

Tadqiqotning usullari. Dissertatsiyada differensial tenglamalar, operatorlar nazariyasi, dinamik sistemalar nazariyasi, ehtimollar nazariyasi usullari qoʻllanilgan.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

cheksiz oʻlchovli fazolarda aniqlangan boshqariluvchan sistemalarning yangi sinfining asimptotik turgʻun va nol boshqariluvchan ekani isbotlandi;

chekli oʻlchovli fazoda aniqlangan, oʻyinchilarning boshqaruviga integral chegaralanish qoʻyilgan differensial oʻyinlarda quvish-qochish masalalari yechildi;

Lorents attraktorining statistik turgʻunligi va dispersiyaning turgʻunligi isbotlangan;

tasodifiy dinamik sistemalarda statsionar oʻlchovning silliqligini isbotlash uchun abstrakt usul ishlab chiqilgan va bu usul yordamida Gauss-Reyni, Manniville-Pomeau akslantirishlarining silliq oʻzgarishi isbotlangan;

tasodifiy dinamik sistemalar uchun deyarli barcha realizatsiyalarda korrelyatsiya funksiyasining kamayish tezligini baholash uchun abstrakt Yong minorasining tasodifiy analogi ishlab chiqilgan va bu usul yordamida kuchsiz giperbolikka ega tasodifiy sistemalar va ularning qoʻzgʻatishlari uchun korrelyatsiya funksiyasining kamayish tezligi uchun baholar olingan;

kritik takrorlanish hodisasi logistik oilaga tegishli akslantirishlarning tasodifiy kompozitsiyalaridan hosil boʻlgan sistema uchun isbotlangan.

Tadqiqot natijalarining ishonchliligi. Natijalarning ishonchliligi differensial tenglamalar, funksional analiz, extimollar nazariyasi, dinamik sistemalar nazariyasi metodlariga asoslangan qa’tiy matematik isbotlar bilan asoslangan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati kuchsiz giperbolik tasodifiy dinamik sistemlarning statistik tadqiq qilish usullarini rivojlantirilishi bilan izohlanadi.

Tadqiqot natijalarining amaliy ahamiyati dinamik sistemalarni tadqiq etishning sifat nazariyasiga tatbiq etilishi bilan izohlanadi.

Tadqiqot natijalarining joriy qilinishi. Dinamik sistemalarda turg‘unlik, ergodiklik va boshqaruv masalalarida olingan natijalar asosida:

tasodifiy dinamik sistemalarda statsionar o‘lchov silliqligini isbotlash uchun ishlab chiqilgan abstrakt usuldan IP-2019-04-1239 raqamli “Transfer operatorlar, limit qonuniyatlar va cheksiz o‘lchovli dinamik sistemalar” mavzusidagi xorijiy loyihada tasodifiy dinamik sistemalar uchun chiziqli javob masalasini yechishda foydalanilgan. (Rijeka Universitetining 2023 yil 21-noyabrdagi ma’lumotnomasi, Xorvatiya). Ilmiy natijalarning qo‘llanishi kuchsiz giperboliklikka ega bo‘lgan tasodifiy sistemalarda chiziqli javob masalasini yechish imkonini bergan.

Lorents attraktorining statistik turg‘unligi va cheksiz o‘lchovli fazolarda aniqlangan boshqaruv sistemalari uchun olingan natijalardan OT-F4-(36+32) raqamli “Matematik fizika va optimal boshqaruv masalalarini yechishning yangi usullarini ishlab chiqish. Toq tartibli xususiy hosilali tenglamalar uchun noklassik boshlang‘ich va chegaraviy masalalar va ularning tadbiqlari” mavzusidagi loyihada optimal boshqaruv masalalarini yechishning yangi usullarini ishlab chiqish va ularni sonli amalga oshirishda kelib chiqadigan chiziqsiz masalalarni yechishda foydalanilgan (O‘zbekiston Milliy universitetining 2023 yil 28-dekabrdagi №04/11-9455-sonli ma’lumotnomasi). Ilmiy natijalarning qo‘llanilishi cheksiz differensial tenglamalar bilan ifodalanuvchi sistemalar uchun optimal boshqaruv sintezini qurish imkonini bergan.

Kuchsiz giperbolik sistemalarda korrelyatsiya funksiyasining kamayish tezligini baholash uchun ishlab chiqilgan abstrakt usuldan xorijiy ilmiy jurnallarda chop etilgan ilmiy maqolalarda korrelyatsiya funksiyasining kamayish tezligini baholash uchun foydalanilgan (Advances in Math, 2023, vol 426, Ann. Probab. 2022, 50 (1) 241 - 303, Commun. Math. Phys. 2021, 385, 905–935). Ilmiy natijalarning qo‘llanishi kuchsiz giperboliklikka ega bo‘lgan sistemalarda korrelyatsiya funksiyasining kamayishi uchun baholar chiqarish imkonini bergan.

Tadqiqot natijalarining aprobatsiyasi. Ushbu tadqiqot natijalari dunyoing yetakchi ilmiy markazlarida tashkil etilgan 9 ta xalqaro ta respublika ilmiy-amaliy anjumanlarda muhokamadan o‘tkazilgan.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya natijalari bo‘yicha jami 25 ta ilmiy ishlar chop etilgan bo‘lib, shundan 13 ta ilmiy maqola O‘zbekiston Respublikasi Oliy attestatsiya komissiyasi tomonidan doktorlik dissertatsiyalari (DSc) asosiy ilmiy natijalarini chop etish tavsiya etilgan ilmiy nashrlarda, jumladan 12 ta ilmiy maqola “Scopus” va “Web of Science Core Collection” ma’lumotlar bazasida indekslangan ilmiy jurnallarda nashr etilgan.

Dissertatsiya kirish, to‘rtta bob, xulosa hamda foydalanilgan adabiyotlar ro‘yxatidan iborat bo‘lib, umimiy hajmi 197 betdan iborat.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida dissertatsiya mavzusining dolzarbligi va zarurati ko‘rsatilgan, ilmiy tadqiqotning respublikada fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi asoslangan, mavzu bo‘yicha xorijiy ilmiy-tadqiqotlar sharhi, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqotning obekti, maqsadi, vazifalari va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertatsiya tuzilishi bo‘yicha ma’lumotlar keltirilgan.

Dissertatsiyaning **“Ziddiyatli va ziddiyatsiz boshqariluvchan sistemalar”** deb nomlanuvchi birinchi bobi asimptotik turg‘ulikka oid natijalarni o‘z ichiga oladi.

Quyidagi taqsimlangan parameterli sistemani qaraymiz

$$w_t = Aw + v, \tag{1}$$

bunda $w(x,t)-$ skalyar holat funksiyasi $x=(x_1,...,x_d)$ vektor va t vaqtga bog‘liq. v boshqaruv parametri va u $v(t,x),$ kabi tanlanadi. A differensial operator bo‘lib tartibi $2m$ ga teng.

(1) tenglama $\Delta = \bar{\Omega}\times\left[0,+\infty\right)$ sohada qaraladi. Bu yerda $\Omega \subset R^d$ chegarasi Γ bo‘lakli silliq bo‘lgan ochiq, chegaralangan to‘plam. Quyidagi chegaraviy shartlar qaraladi:

$$w(x,0)=w_0(x),\ x\in\Omega,\ Mw=0,\ x\in\Gamma;\ M=(M_1,M_2,...,M_m). \tag{2}$$

bunda $M_1,M_2...,M_m$ tartibi $2m$ dan kichik bo‘lgan differensial operatorlar.

Boshqaruv parametri $v\in L_2(\Delta)$ va biror $v^0>0$ uchun quyidagi

$$\left|v(x,t)\right|\le v^0,\ x\in\Omega,\ t\ge 0, \tag{3}$$

Tengsizlikni qanoatlantirsa, v joiz boshqaruv deyiladi.

Boshqaruv masalasida (1) tenglamaning (2) shartlarni qanoatlantiruvchi $w(x,t)\in W^{2m,1}\left(\Delta\right)$ yechimi mavjud bo‘ladigan va bu yechim biror $T, T>0$ va deyarli barcha $x\in\Omega$ uchun $w(x,T)=0$ shartni qanoatlantiradigan $v(x,t)$ funksiyani topish talab etiladi. Bunday joiz boshqaruvlarga kafolatlangan boshqaruv, T esa kafolatlangan o‘tish vaqti deyiladi. Barcha joiz boshqaruvlar bo‘yicha olingan minimum $T^*=\inf T$ optimal o‘tish vaqti deyiladi, T^* erishadigan boshqaruv optimal boshqaruv deyiladi.

(1) tenglamadagi A operatorning xos funksiyalaridan tuzilgan $\left\{\varphi_k(x)\right\}$ sistema to‘la bo‘lsin deb faraz qilamiz. Hamda $w(x,t)$ va $v(x,t)$ uchun mos ravishda quyidagi:

$$w(x,t)=\sum_k q_k(t)\varphi_k(x),\quad v(x,t)=\sum_k \tilde{u}_k(t)\varphi_k(x)$$

yoyilmalar o‘rinli bo‘lsin. U holda $w(x,t)$, $v(x,t)$ juftlk (1) – (3) masalaning yechimi bo‘lishi q_k va $\tilde{u}_k$ funksiyalarning

$$\dot{q}_k + \lambda_k q_k = \tilde{u}_k \tag{4}$$

cheksiz sistemani

$$q_k(0) = q_k^0 = \int_\Omega w_0(x)\varphi_k(x)dx. \tag{5}$$

boshlang‘ich shartlar bilan qanoatlantirishiga teng kuchli. Bu holda (3) shart

$$\sup_{x\in\Omega}\left|\sum \varphi_k(x)\tilde{u}_k(t)\right| \le v^0.$$

shartga o‘tadi. Yuqoridagi shartni qanoatlirish uchun $\tilde{u}_k$ funksiyalarni

$$\sum_k \Phi_k \left|\tilde{u}_k(t)\right| \le v^0, \tag{6}$$

tengsizlik bajariladigan qilib tanlash yetarli, bu yerda $\Phi_k = \sup\limits_{x\in\Omega}\left|\varphi_k(x)\right|$. F. L. Chernouskoning optimal boshqaruvni dekompozitsiya qilish usuli bilan izlash haqidagi ilmiy ishida (6) tengsizlikni $\left|\tilde{u}_k\right| \le U_k$, $k \ge 1$, shartlar bilan almashtiradi, bunda U_k lar $\sum\limits_k \Phi_k U_k \le v^0$ tengsizlkni qanoatlantiradigan qilib tanlangan. Tabiyki, oxirgi shartdan (6) tengsizlik kelib chiqadi, ammo teskarisi o‘rinli emas. (4)-(6) sistema uchun nol boshqaruv masalasini ω-boshqaruv masalasi deb ataymiz. Chiziqli almashtirishlar yordamida (4) sistema va (6) shartni quyidagi ko‘rinishga keltiramiz

$$\dot{y}_k + \lambda_k y_k = u_k, \sum_k \left|u_k\right| \le 1. \tag{7}$$

Ixtiyoriy musbat n soni uchun ushbu

$$\dot{y}_j + \lambda_j y_j = u_j, U = \left\{u \in R^n \middle| \left|u_1\right| + \left|u_2\right| + ... + \left|u_n\right| \le 1\right\} \tag{8}$$

sistemani qaraymiz. (7)-(8) masala chekli o‘lchovli fazoda aniqlangan va uning uchun boshqaruv sohasi uchlari

$$e_j^\sigma = \sigma(\delta_{j1},...,\delta_{j2},...,\delta_{jn}), \quad \sigma \in \{-1,+1\}$$

bo‘lgan n o‘lcholvi oktaedrdan iborat, bu yerda δ_{ji} Kroneker deltasi.

L.S. Pontryaginning maksimum prinsipini qo‘llaymiz: barcha $k \ne l$ uchun $\lambda_k \ne \lambda_l$ ekanidan (7)-(8) tezkorlik masalasining yechimi mavjud va yagona, hamda bo‘lakli o‘zgarmas. Xos sonlar nomanfiyligidan ixtiyoriy nuqtadan boshlangan optimal boshqaruv mavjud bo‘ladi va optimal trayektoriyalar

$$\dot{y}_j = -\lambda_j y_j \pm 1, \ \dot{y}_s = -\lambda_s y_s \ \text{ bu yerda } s \ne j, j \le n$$

tenglamalar sistemasining yechimi bo‘ladi. Bu tengliklarda $y_j(t) = 0$ tenglamani y_j^0 ga nisbatan yechsak, barcha $j = 1,2,...,n-1$ lar uchun

$$T_j = \frac{1}{\lambda_j}\ln\left(\lambda_j\left|y_j^0\right| + e^{\lambda_j T_{j-1}}\right) = \frac{1}{\lambda_j}\ln\left(\lambda_j\left|y_n^0\right|e^{-\lambda_n T_{j-1}} + 1\right) + T_{j-1}; \quad T_0 = 0, \quad (9).$$

rekurrent munosabatlar hosil bo‘ladi. Shu sababli optimal trayektoriya uchun $y_j(T_j)=0$ o‘rinli bo‘ladi. Bu yerdan optimal boshqaruv oson topiladi:

$$u_j(t)=\begin{cases}-sgny_j^0 \text{ agar}\left[T_{j-1},T_j\right), \\ 0 \ \text{ agar } t\in\left[0,T_j\right]\setminus\left[T_{j-1},T_j\right).\end{cases}$$

1-teorema. *Agar (9) munosabatning $n\to\infty$ dagi limiti $T_*(y^0)$ mavjud bo‘lsa, bu limit (4)-(6) ω – boshqaruv masalasi uchun optimal o‘tish vaqti bo‘ladi. Bundan tashqari quyidagi tengsizlik o‘rinli*

$$T_*\le\left|y_p^0\right|+\sum_{k=p+1}^{\infty}\left|y_k^0\right|e^{-\lambda_k\left|y_p^0\right|}.$$

bu yerda p soni: $y_p^0\neq 0$ shartni qanoatlantiruvchi eng kichik natural son.

Quyidagi standard belgilashlarni kiritamiz:

$$\ell^2=\{y=(y_1,y_2,\ldots)\,|\sum\nolimits_{n\ge1}y_n^2<\infty\}.$$

Ma’lumki, ℓ^2 fazosida $\|y\|_2^2=\sum\nolimits_{n\ge1}y_n^2$ norma aniqlangan. Banax fazosidagi oddiy differensial tenglamaga misol sifatida quyidagi sistemani qaraymiz:

$$\dot{y}=Ay,\quad y(0)=y_0,$$

bu yerda $y_0=\{y_{n,0}\}_{n\in\mathbb{N}}$ va $A:\ell^2\to\ell^2$ esa $Ay=\{\lambda y_n+y_{n+1}\}_{n\in\mathbb{N}}$, tenglik bilan aniqlangan chiziqli operator. Endi ushbu tenglama uchun turg‘unlik va boshqaruv masalalarini tadqiq qilamiz. Xususan, boshqaruv funksiyasini oshkor ko‘rinishda quramiz. Shuningdek, $f\in L^2([0,T],\ell^2)$, ya’ni $\|f\|_{L^2}^2=\int_0^T\|f(t)\|_2^2\,dt<+\infty$[1] uchun ushbu

$$\dot{y}=Ay+f,\quad y(0)=y_0, \tag{10}$$

bir jinsli bo‘lmagan tenglama uchun Koshi masalasini ko‘rib chiqamiz.

Ushbu

$$y(t)=e^{tA}y_0+e^{tA}\int_0^t e^{-sA}f(s)ds$$

ko‘rinishda aniqlangan $y:[0,T]\to\ell^2$ funksiya (10) tenglamaning “yumshoq” yechimi deyiladi. Bu yerda integrallash komponentlar bo‘yicha tushuniladi.

1-tasdiq. *Barcha $f\in L^2([0,T],\ell^2)$ va $y_0\in\ell^2$ uchun $y\in C([0,T],\ell^2)$ bo‘ladi.*

2-tasdiq. *Aytaylik $y(t)$ funksiya (10) masalaning $y_0\in\ell^2$ boshlang‘ich sharni qanoatlantiruvchi yechimi bo‘lsin. (10) sistema asimptotik turg‘un bo‘lishi uchun $\lambda\le-1$ bo‘lishi zarur va yetarli. Bundan tashqari, har qanday $y_0\in\ell^2$ va $t\in\mathbb{R}$ uchun $\|e^{tA}y_0\|_2\le e^{(1+\lambda)t}\left\|y_0\right\|_2$ tengsizlik o‘rinli bo‘ladi.*

Aytaylik $\rho>0$ bo‘lsin. Agar $f:\mathbb{R}\to\ell^2$ boshqaruv funksiyasi uchun

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[1] Beppo-Levi teoremasiga ko‘ra ushbu normaning $\|f\|_{L^2}^2=\sum\nolimits_{n\ge1}\int_0^T|f_n(t)|^2\,dt$ ga mos kelishini qayd etamiz.

$$\| f \|_{L^2}^2 = \int_0^T \| f(t) \|_2^2 \, dt \le \rho^2$$
tengsizlik o‘rinli bo‘lsa, u holda bunday boshqaruv funksiyasi joiz boshqaruv deb ataladi.

Agar shunday $f : \mathbb{R} \to \ell^2$ joiz boshqaruv va $T = T(f) \in \mathbb{R}_+$ mavjud bo‘lib, (10) sistemaning yechimi $y(T) = 0$ tenglikni qanoatlantirsa, u holda (10) sistemani $y_0 \in \ell^2$ nuqtadan “*nol-boshqariluvchan”* deyiladi.

Agar shunday $\delta = \delta(\rho) > 0$ topilib, (10) sistema $\|y_0\| \le \delta$ shartni qanoatlantiruvchi ixtiyoriy $y_0 \in \ell^2$ nuqtadan nol-boshqariluvchan bo‘lsa, u holda (10) sistema *local nol-boshqariluvchan* deyiladi.

Agar sistema ixtiyoriy $y_0 \in \ell^2$ nuqtadan nol-boshqariluvchan bo‘lsa, u holda (10) sistema *global nol-boshqariluvchan* deyiladi.

2-teorema. (i) *Barcha* $\lambda \in \mathbb{R}$ *uchun* (10) *sistema local nol-boshqariluvchan bo‘ladi;*

(ii) (10) *sistema global nol-boshqariluvchan bo‘lishi uchun* $\lambda \le -1$ *bo‘lishi zarur va yetarli.*

(iii) *Agar* $\lambda < -1$ *bo‘lsa, shunday* κ *topilib, ixtiyoriy* $y_0 \in \ell^2$ *nuqtadan 0 ga o‘tish vaqti* τ *uchun* $\tau \le \|y_0\|_2^4 / \kappa\rho^4$ *tengsizlikni qanoatlantiradi.*

Ta’kidlash joizki, $-1 < \lambda < 0$ uchun $t \to +\infty$ da $\|y(t)\|_2 \to \infty$ bo‘lgan yechimlar mavjud, ya’ni $y = 0$ yechim Lyapunov ma’nosida turg‘un emas.

1-izoh. $\dot{y} = Ay$, $y(0) = y_0 \in \ell^\infty$ sistemani qaraymiz, bu yerda
$$\ell^\infty = \{ y = (y_1, y_2, \ldots) \mid \sup_{n \in \mathbb{N}} | y_n | < \infty \}.$$
Ravshanki, $e = (1,1,1\ldots) \in \ell^\infty$ – e^{tE} ning e^t xos qiymatiga mos xos vektor. Demak, 0-yechim Lyapunov ma’nosida turg‘un, lekin u asimptotik turg‘un emas.

Boshqaruvchanlikni isbotlash uchun Gram operatorlaridan foydalanamiz, ya’ni ixtiyoriy $\tau \in \mathbb{R}$ uchun quyidagi operatorni qaraymiz
$$W(\tau) = \int_0^\tau e^{-sA} \cdot e^{-sA^*} ds.$$
W operatorning o‘z-o‘ziga qo‘shma ekanligi va uning teskarisining normasini baholash quyidagi standart lemmadan kelib chiqadi:

1-lemma. *Aytaylik* $L : \mathcal{H} \to \mathcal{H}$, $(\mathcal{H}, \| \cdot \|)$ *Hilbert fazosida aniqlangan o‘z-o‘ziga qo‘shma operator hamda barcha* $x \in L$ *uchun* $\| Lx \| \ge \kappa \| x \|$ *tengsizlik o‘rinli bo‘ladigan* $\kappa > 0$ *mavjud bo‘lsin. U holda* L *teskarilanuvchi va* $\| L^{-1} \| \le \kappa^{-1}$ *tengsizlik bajariladi.*

Endi quyidagi sanoqli differensial tenglamalar bilan aniqlangan
$$\dot{x}_i = A_i x_i + u_i - v_i, \; x_i(0) = x_{i0} \in \mathbb{R}^{d_i}, \; i = 1, 2, ..., \qquad (11)$$
differensial o‘yinni qaraymiz. Bu yerda ixtiyoriy $i \in \mathbb{N}$ uchun A_i matritsalar $d_i \times d_i$ tartibli, biror d uchun $2 \le d_i \le d$ shartni qanoatlantiradi, u_i va v_i – mos

ravishda quvlovchining va qochuvchining boshqaruv parametrlari. Boshqaruv parametrlari ma'lum bir cheklovlarni qanoatlantiruvchi funksiyalar deb qaraladi. Ixtiyoriy $x_0 = (x_{10}, x_{20}, \ldots,) \in \ell^2$ uchun $\| x \|^2 = \sum_{i=0}^{\infty} \| x_{i0} \|^2 < +\infty$ normani aniqlaymiz. Bu yerda $\| x_{i0} \|$ – $x_{i0} \in \mathbb{R}^{d_i}$ Yevklid normasi. Shuningdek $A = \mathrm{diag}(A_1, A_2, \ldots)$ – operator uchun e^{tA} operatorni aniqlaymiz va (11) bilan bir qatorda bir jinsli bo'lmagan tenglamani ham qaraymiz, bu yerda $w_i : \mathbb{R} \to \mathbb{R}^{d_i}$ lokal integrallanuvchi funksiyalar. Quyidagi

$$\dot{x}_i = A_i x_i + w_i, \ x_i(0) = x_{i0} \in \mathbb{R}^{d_i}, \ i = 1, 2, \ldots, \tag{12}$$

Koshi masalasining yechimi mavjudligini isbotlash kerak. $C([0,T];\ell^2)$ uzluksiz funksiyalar fazosidan biror $T > 0$ uchun (12) masalaning shunday yechimini izlaymizki, $x : [0,T] \to \ell^2$ ning $x_i(\cdot)$ koordinatalari deyarli hamma joyda differensiallanuvchi bo'lsin.

1-ta'rif. *Agar singulyar bo'lmagan $\{P_i\}_{i \in \mathbb{N}}$ matritsalar oilasi va o'zgarmas $C \geq 1$ mavjud bo'lib, $\| P_i \| \cdot \| P_i^{-1} \| \leq C$ va $P_i A_i P_i^{-1}$ barcha $i \in \mathbb{N}$ uchun Jordan normal ko'rinishdagi matritsalar bo'lsa, $\{A_i\}_{i \in \mathbb{N}}$ matritsalar oilasi tekis normallanuvchi deyiladi.*

Tayinlangan $\theta > 0$ uchun

$$\sum_{i=1}^{\infty} \int_0^T \| w_i(s) \|^2 \, ds \leq \theta^2$$

cheklovni qanoatlantiruvchi $w_i(\cdot) \in \mathbb{R}^{d_i}$, $0 \leq t \leq T$, $i = 1, 2, \ldots$, o'lchovli koordinatalarga ega barcha $w(\cdot) = \left(w_1(\cdot), w_2(\cdot), \ldots \right)$, $w : [0,T] \to \ell^2$ funksiyalar to'plamini $B(\theta)$ bilan belgilaymiz.

2-ta'rif. *$B(\theta)$ joiz boshqaruv funksiyalar to'plami deb ataladi.*

3-**teorema.** *Agar $\{A_i\}$, $i = 1, 2, \ldots$, tekis normallanuvchi matritsalar oilasida har bir A_i, $i = 1, 2, \ldots$, matritsaning xos qiymatlari haqiqiy qismlari manfiy bo'lsa, u holda (12) sistema ixtiyoriy $w \in B(\theta)$, $\theta > 0$ va ixtiyoriy $x_0 \in \ell^2$ uchun yagona yechimga ega bo'ladi. Shuningdek, $x(t) = (x_1(t), x_2(t), \ldots)$ yechimning tegishli komponentlari*

$$x_i(t) = e^{tA_i} x_{i0} + \int_0^t e^{(t-s)A_i} w_i(s) ds, \quad i \in \mathbb{N},$$

ko'rinishda aniqlanadi.

4-teorema. 3-*teorema shartlari bajarilganda (12) sistema global asimptotik turg'un va global nol-boshqariluvchan bo'ladi. Tezkorlik masalasining yechimi mavjud va uni oshkor ravishda qurish mumkin.*

Endi (11) differensial o'yinga qaytamiz. Berilgan $\rho, \sigma > 0$ sonlari uchun $u(\cdot) \in B(\rho)$ $(v(\cdot) \in B(\sigma))$ funksiyani quvlovchining (qochuvchining) joiz boshqaruvi deyiladi.

3-ta'rif. *Har bir* $v \in \ell^2$ *uchun koordinatalari* $u_k(t) = v_k(t) + \omega_k(t)$, $\omega \in B(\rho - \sigma)$ *bo'lgan* $u:[0,T] \times \ell^2 \to \ell^2$ *quvlovchining joiz boshqaruvi quvlovchining strategiyasi deyiladi.*

5-teorema. *Faraz qilaylik,* $\rho > \sigma$ *va 3-teorema shartlari bajarilsin. U holda qochuvchining ixtiyoriy* v *joiz boshqaruvi uchun quvlovchining shunday* u *strategiyasi va* $\vartheta_1 > 0$ *mavjudki, (11) differensial o'yinning yechimi biror* $0 \le \tau \le \vartheta_1$ *uchun* $x(\tau) = 0$ *tenglikni qanoatlantiradi, ya'ni (11) o'yinni* ϑ_1 *vaqtda yakunlash mumkin.*

Quyigadi chiziqli tenglama bilan tavsiflanuvchi differesial o'yinni qaraymiz

$$\dot{x}_i = -\lambda_i x_i + v - u_i, \quad x_i(0) = x_i^0 \in \mathbb{R}^n, \quad i = 1, 2, \ldots, m, \tag{13}$$

bu yerda $u_1, \ldots, u_m$ – quvlovchilarning, v – qochuvchining boshqaruv parametrlari va $\lambda_i > 0$, $x_i \in \mathbb{R}^n$, $n \ge 2$, $x_i^0 \ne 0$, $i = 1, \ldots, m$.

5-ta'rif. *Ushbu*

$$\int_0^\infty |u_i(t)|^2\, dt \le \rho_i^2, i = 1, \ldots m; \int_0^\infty |v(t)|^2\, dt \le \sigma^2, \tag{14}$$

integral cheklovlarni qanoatlantiruvchi $u_i(t)$ *va* $v(t)$, $t \ge 0$ *o'lchovli funksiyalar, mos ravishda,* $i-$*chi quvlovchi va qochuvchining boshqaruvlari deb ataladi.*

6-teorema. *Agar*

$$\rho_1^2 + \ldots + \rho_m^2 \le \sigma^2,$$

bo'lsa, u holda (13)-(14) o'yinda qochish mumkin, ya'ni qochuvchining oshkor ko'rinishda qurilgan V strategiyasi mavjud bo'lib, quvlovchilarning har qanday boshqaruvi uchun barcha $i = 1, \ldots, m$ *va* $t \ge 0$ *uchun* $x_i(t) \ne 0$ *munosabat bajariladi.*

Dissertatsiyaning **"Lorents sistemasi uchun statistik xossalarning turg'unligi"** deb nomlangan ikkinchi bobida Lorents sistemasi uchun markaziy limit teoremada dispersiyaning uzluksizligi va sistema invariant o'lchovining uzluksizligi o'rganiladi. Lorents atmosferadagi konveksiya uchun soddalashtirilgan model sifatida quyidagi tenglamalar sistemasini taklif qilgan.

$$\dot{x} = -10x + 10y, \quad \dot{y} = 28x - y - xz, \quad \dot{z} = -\frac{8}{3}z + xy. \tag{15}$$

Lorents tomonidan o'tkazilgan sonli tahlil bu sistemaning boshlang'ich shartlarga nisbatan yuqori sezgirligini va davriy bo'lmagan "g'aroyib" attraktorga ega ekanligini ko'rsatgan. Shunga o'xshash oqimlarning qat'iy matematik asoslari ilmiy ishlarda geometrik Lorents oqimini kiritish orqali boshlangan. Bugungi kunda X_0 vektor maydon bilan aniqlanadigan geometrik Lorents attraktori, C^1 topologiyada barqaror ekanligi yaxshi ma'lum. Bu agar X_ε vector maydonlar X_0 ga C^1 topologiyada yaqin bo'lsa, u holda bunday vector maydonlar ham Lorents attraktoriga o'xshash attraktorga ega bo'lishini bildiradi. Aniqroqg'i, X_0 ga C^1 topologiyasida yetarlicha yaqin bo'lgan X_ε vektor maydonlar Puankare kesimida invariant, qisqartuvchi $\mathcal{F}_\varepsilon$ foliyatsiyalarga ega va ular g'aroyib attraktorlar hosil qiladi. Aniqroq qilib aytganda, $\mathbb{R}^3$ ichida Λ, ya'ni X_0 ning attraktori joylashgan ochiq soha U va X_0 ning C^1 topologiyasidagi ochiq soha $\mathcal{U}$ mavjud bo‘lib, har bir $X_\varepsilon \in \mathcal{U}$ ning trayektoriyalari U ichida yotib, maksimal invariant to‘plam $\Lambda_{X_\varepsilon} = \bigcap_{t\geq 0} X_\varepsilon^t(U)$ tranzitiv va X_ε oqimiga nisbatan invariantdir.

$C^{1+\alpha}$ **turg‘un foliatsiyaga ega geometrik Lorents oqimi.** $X_0 : \mathbb{R}^3 \to \mathbb{R}^3$ vektor maydoni bo‘lib, 0 uning muvozanat nuqtasi bo‘lsin. Faraz qilaylik, X_0 quyidagilarni qanoatlantirsin:

• $DX_0(0)$ differensiali uchta haqiqiy $\lambda_2 < \lambda_3 < 0 < \lambda_1$, $\lambda_1 + \lambda_3 > 0$ hos sonlarga ega bo‘lsin (Lorenz tipidagi maxsuslik) va $\lambda_1 + \lambda_2 < \lambda_3$. Bundan tashqari,

$$\Sigma := \left\{ (x, y, 1) \,|\, -\frac{1}{2} \leq x, y \leq \frac{1}{2} \right\}$$

Puankare kesimi bo‘lib, Σ va 0 nuqtadagi lokal turg‘un ko‘pxillikining kesishmasi $\Gamma = \left\{ (0, y, 1) \,|\, -\frac{1}{2} \leq y \leq \frac{1}{2} \right\}$ bo’lsin va Σ ni mos ravishda quyidagi ikki qismga ajratsin:

$$\Sigma^+ = \{(x, y, 1) \in \Sigma : x > 0\} \text{ va } \Sigma^- = \{(x, y, 1) \in \Sigma : x < 0\}.$$

Puankare asklantirishi $F : \Sigma \to \Sigma$ orqali $\Sigma^\pm$ sohalarning tasvirlari $S^\pm$ egri uchburchaklar shaklida bo‘lsin. Bunda $(\pm 1, 0, 1)$ dan tashqari, va $\mathcal{F} = \{(x, y, 1) \in \Sigma \,|\, x = \text{const}\}$ to‘plamidagi har bir kesma, Γ ni hisobga olmaganda, $\{(x, y, 1) \in S^\pm \,|\, x = \text{const}\}$ segmentga akslanadi. $S^\pm$ va qaytish vaqti $\tau : \Sigma \setminus \Gamma \to \mathbb{R}$ quyidagicha bo‘lsin: $\tau(x, y, 1) = -\frac{1}{\lambda_1} \log |x|$.

• Oqim Σ ni silliq tarzda $S^\pm$ ga akslantiradi. Bunda Puankare akslantirishi $F(x, y) = X_0^{\tau(x,y,1)}((x, y, 1), t)$ quyidagi ko‘rinishga ega bo‘lsin:

$$F(x, y) = (T(x), g(x, y)),$$

bu yerda $T : I \to I$, $I := [-1/2, 1/2]$, va $x = 0$ uzilish nuqtasi, hamda $T(0^+) = -1/2$ va $T(0^-) = 1/2$;

• T o‘suvchu va $C^1(I \setminus \{0\})$ sinfga tegishli;

• $\lim_{x \to 0^\pm} T'(x) = +\infty$;

• $\frac{1}{T'}$ funsiya $I_1 := [-\frac{1}{2}, 0]$ va $I_2 := [0, \frac{1}{2}]$ intervallarda α-Ho‘lder uzluksiz bo‘lsin, bu yerda $0 < \alpha \leq 1$;

• Shunday $C > 0$ va $\theta > 1$ topilib, ixtiyoriy $n \in \mathbb{N}$ uchun $(T^n)'(x) \geq C\theta^n$;

• T tranzitiv (ya’ni hamma yerda zich trayektoriyaga ega nuqtasi bor);

• g akslantirsh $\mathcal{F}$ ni saqlaydi va tekis qisqartuvchi, ya’ni shunday $K > 0$ va $0 < \rho < 1$ topilib, ixtiyoriy $\gamma \in \mathcal{F}$ va $\xi_1, \xi_2 \in \gamma$ va $n \geq 1$ uchun

$$\operatorname{dist}(F^n(\xi_1),F^n(\xi_2))\le K\rho^n\operatorname{dist}(\xi_1,\xi_2).$$

Universal p-cheklangan variatsiya. Agar $p\ge 1$ uchun

$$V_p(f):=\sup_{-1/2\le x_0<\ldots<x_n\le 1/2}\left(\sum_{i=1}^{n}\left|f(x_i)-f(x_{i-1})\right|^p\right)^{1/p}<+\infty.$$

tengsizlik bajarilsa, u holda $f:I\to\mathbb{R}$ funksiya universal p variatsiyasi cheklangan funksiya deyiladi. Universal p variatsiyasi checklangan funksiyalar $UBV_p(I)$ bilan belgilanadi. Endi $p:=\dfrac{1}{\alpha}$ deb belgilaymiz. X_0 ning C^1 qo‘zg‘atishlari oilasi $\mathcal{X}$ ni quyidagicha aniqlaymiz: $\mathbb{R}^3$ da X_0 ning attraktori bo‘lgan Λ ni o‘z ichiga olgan ochiq soha U va $\mathcal{X}$ ni o‘z ichiga olgan X_0 ning ochiq atrofi $\mathcal{U}$ mavjud bo‘lib, quyidagilar tasdiqlar o‘rinli bo‘lsin:

1. har bir $X_\varepsilon\in\mathcal{X}$ uchun maksimal musbat invariant to‘plam $\Lambda_{X_\varepsilon}\subset U$ giperbolik maxsuslikka ega bo‘lgan attraktor;

2. har bir $X_\varepsilon\in\mathcal{X}$ aniqlagan oqim uchun Σ Puankare kesimi bo‘lib, qaytish vaqti τ_ε va Puankare akslantirishsi F_ε mavjud;

3. har bir $X_\varepsilon\in\mathcal{X}$ uchun $F_\varepsilon:\Sigma\to\Sigma$ qisqartuvchi, qisqartish koeffitsienti ε ga bog‘liq bo‘lmagan $C^{1+\alpha}$ invariant $\mathcal{F}_\varepsilon$ foliatsiyaga ega;

4. $F_\varepsilon:\Sigma\to\Sigma$ quyidagicha aniqlangan[1]

$$F_\varepsilon(x,y)=(T_\varepsilon(x),g_\varepsilon(x,y));$$

5. $T_\varepsilon:I\to I$ akslantirish ikki tarmoqli va uzilish nuqtasi O_ε ga ega bo‘lgan C^1 tranzitiv va har bir tarmog‘i kengaytiruvchi bo‘lib, $T(O^+)=-\dfrac{1}{2}$, $T_\varepsilon(O^-)=\dfrac{1}{2}$ va $x\to O^{\pm}{}_\varepsilon$ da $\lim\limits_{\varepsilon\to 0}T'_\varepsilon(x)=+\infty$;

6. Har qanday $\eta_0>0$ uchun $H(\eta_0)\subset I$ interval mavjud bo‘lib, barcha $0\le\eta<\eta_0$ uchun $\{O_\varepsilon,0\}\in H,\ |H|=2|O_0|<\eta;$ va

$$d(T,T_\varepsilon):=\sup_{x\in H^C}\{|T_\varepsilon(x)-T(x)|+|T'_\varepsilon(x)-T'(x)|\}\le C\eta,$$

bu yerda $H^c:=I\setminus H$;

7. ε va x bo‘yicha shunday $C>0$ va $\theta>1$ o‘zgarmaslar mavjud bo‘lib, $T_\varepsilon^n(x)\ge C\theta^n$ tengsizlik O_ε uzilish nuqtasidan tashqari barcha nuqtalarda bajariladi;

8. ε ga bog‘liq bo’lmagan $W>0$ mavjud bo‘lib, quyidagi

$$\max_{i\in 1,2}V_{p|_{I_i,}}\left(\frac{1}{T'_\varepsilon}\right)\le W,$$

tengsizlik o‘rinli, bu yerda I_i, T_ε ning monotonlik intervali va $V_p(\cdot)$ p-variatsiyadir;

---

[1] c) bandiga ko‘ra, g $C^{1+\alpha}$ va turg’un barglarda bir tekis qisqaruvchi funksiyadir.

9. Ixtiyoriy $n>0$ uchun $\mathcal{P}^{(n)} := \vee_{j=0}^{\ell-1} T^{-j}(\mathcal{P}_\varepsilon)$ bo‘lsin, bu yerda $\mathcal{P}_\varepsilon = \{I_1, I_2,\}$ bo‘lib, ε ga bog‘liq bo‘lmagan shunday qiymat $\delta_n > 0$ mavjudki, $\mathcal{P}^{(n)}$ da J uchun $\min|J| \geq \delta_n$;

10. Qaytish vaqti $\tau_\varepsilon : \Sigma \setminus \Gamma_\varepsilon \to \mathbb{R}$ va ixtiyoriy $X \in \mathcal{X}$ uchun shunday $C>0$ topilib,

$$\tau_\varepsilon(\xi) \leq -C \log|\pi_\varepsilon(\xi) - O_\varepsilon|,$$

tengiszlik o‘rinli bo‘ladi. Bu yerda π akslantirish $\mathcal{F}_\varepsilon$ ning barglari bo‘ylab I ga proyeksiyalashdir.

Quyidagi fazolar 1985 yilda G. Keller tomonidan kiritilgan.

Banach fazosi. $S_\rho(x) := \{y \in I : |x-y| < \rho\}$ va $f: I \to \mathbb{R}$ bo‘lib, I intervalda aniqlangan funksiya bo‘lsin. Quyidagicha belgilashlarni kiritamiz:

$$\mathrm{osc}(f,\rho,x) := \{|f(y_1) - f(y_2)| : y_1, y_2 \in S_\rho(x)\},$$

va

$$\mathrm{osc}_1(f,\rho) = \|\mathrm{osc}(f,\rho,x)\|_1,$$

bu yerda “essential supremum” $I \times I$ da ikki o‘lchamli Lebeg o‘lchovi bo‘yicha olinadi, $\|\cdot\|_1$ esa I intervalda Lebeg o‘lchovi bo‘yicha L^1 normani bildiradi. Ixtiyoriy $p>1$ uchun $BV_{1,1/p} \subset L^1$ ekanligi uchun quyidagicha normani aniqlaymiz:

$$\|f\|_{1,1/p} = V_{1,1/p}(f) + \|f\|_1,$$

bu yerda

$$V_{1,1/p}(f) = \sup_{0<\rho\leq\rho_0} \frac{\mathrm{osc}_1(f,\rho)}{\rho^{1/p}}.$$

$\rho_0 > 0$ berilgan son. Qayd etish lozimki, $V_{1,1/p}(\cdot)$ ρ_0 ga bog‘liq. $BV_{1,1/p}$ Banach fazosi ekanligi ma’lum. Bundan tashqari, $BV_{1,1/p}$ ning birlik shari L^1 da kompakt.

Sinay-Ruelle-Bowen (SRB) o‘lchovi invariant o‘lchov bo‘lib:

• uning barcha Lyapunov ko‘rsatkichlari nolga teng emas;

• u sistemaning noturg‘un ko‘pxilliklari bo‘ylab tabiiy o‘lchovga (masalan, Lebeg o‘lchoviga) nisbatan absolyut uzluksiz shartli o‘lchovlarga ega.

Aytalik, V interval 0 ning biror atrofi bo‘lsin. $(X_\varepsilon)_{\varepsilon\in V}$ oqimlar oilasida biror $\mathfrak{T}$ topologiya kiritilgan bo‘lsin. Har bir X_ε yagona SRB o‘lchoviga ega bo‘lsin deb faraz qilamiz. $(X_\varepsilon)_{\varepsilon\in V}$ oilasi statistik turg‘un deb ataladi, agar $\varepsilon \mapsto \mu_\varepsilon = \mu_{X_\varepsilon}$ zaif $*$-topologiyada $\varepsilon = 0$ da uzluksiz bo‘lsa, ya’ni

$$\lim_{\varepsilon\to 0} \int f d\mu_\varepsilon = \int f d\mu_0,$$

tenglik har qanday $f: \mathbb{R}^3 \to \mathbb{R}$ uzluksiz funksiyasi uchun o‘rinli bo‘lsa. Statistik turg‘unlik diskret vaqtli dinamik sistemalar uchun ham xuddi shunday aniqlanadi.

7-teorema *$X \in \mathcal{X}$ bo‘lsin. Unda*

• *X_ε oqim μ_ε invariant ehtimollik SRB o‘lchovga ega.*

• *har qanday uzluksiz φ: $\mathbb{R}^3 \to \mathbb{R}$ uchun quyidagi tenglik o‘rinli:*

$$\lim_{\varepsilon\to 0}\int \varphi d\mu_\varepsilon = \int \varphi d\mu;$$

ya'ni $\mathcal{X}$ oila statistik turg'un. Bu yerda μ o'lchov X_0 oqimning SRB o'lchovi.

$\psi:\mathbb{R}^3\to\mathbb{R}$ bo'lsin. $\varepsilon\geq 0$ uchun quyidagi funksiyani aniqlaymiz:

$$\Psi_\varepsilon(\xi):=\int_0^{\tau_\varepsilon(\xi)}\psi(X_\varepsilon(t))dt, \text{ bu yerda } \xi:=(x,y,1)\in\Sigma\setminus\Gamma_\varepsilon.$$

8-*teorema* $\psi:\mathbb{R}^3\to\mathbb{R}$ *Ho'lder uzluksiz bo'lsin.*

• *Unda*

$$\frac{1}{\sqrt{t}}\left(\int_0^t \psi\circ X_\varepsilon(s)ds - t\int\psi d\mu_\varepsilon\right)dt \xrightarrow{\text{law}} \mathcal{N}(0,\sigma^2_{X_\varepsilon})$$

va

$$\sigma^2_{X_\varepsilon}(\psi_\varepsilon)=\sigma^2_{F_\varepsilon}(\Psi_\varepsilon)/\int\tau_\varepsilon d\mu_{F_\varepsilon}.$$

• *Bundan tashqari,* $\psi\in C^1$ *uchun quyidagi tenglik o'rinli:*

$$\lim_{\varepsilon\to 0}|\sigma^2_{X_\varepsilon}(\psi)-\sigma^2_X(\psi)|=0.$$

9-teorema *T_ε akslantirish $\mathcal{X}$ oilaga mos bir o'lchovli akslantirishlar oilasi bo'lib, μ o'lchov X_0 ning SRB o'lchovi bo'lsin. U holda,*

• *T_ε akslantirish $\sigma^2_{T_\varepsilon}$ dispersiya bilan markaziy limit teoremani qonoatlantiradi; ya'ni, $\psi: I\to\mathbb{R}$, $\psi\in BV_{1,1/p}$ bo'lib, $\int\psi d\bar{\mu}_\varepsilon=0$, bo'lsa, bu yerda $\bar{\mu}_\varepsilon$ absolyut uzluksiz T_ε ($\varepsilon\geq 0$) invariant o'lchov, u holda:*

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\psi(T_\varepsilon^i x)\xrightarrow{\text{law}}\mathcal{N}(0,\sigma^2_{T_\varepsilon})$$

o'rinli bo'ladi. Shuningdek, $\sigma^2_{T_\varepsilon}>0$ tengsizlik faqat va faqat $\psi\neq c+\nu\circ T-\nu$, $\nu\in BV_{1,1/p}$, $c\in\mathbb{R}$ bo'lganda o'rinli bo'ladi.

• *$\psi\in BV_{1,1/p}$ bo'lib, $\|\cdot\|_\infty$ normalar bo'yicha tekis chegaralangan va $\varepsilon\to 0$ da $\int|\psi_\varepsilon-\psi|dx\to 0$ bo'lsa, u holda quyidagi tenglik o'rinli bo'ladi:*

$$\lim_{\varepsilon\to 0}|\sigma^2_{T_\varepsilon}(\psi)-\sigma^2_T(\psi)|=0.$$

Dissertatsiyaning uchinchi bobi **"Kuchsiz kengaytiruvchi tasodifiy sistemalar uchun chiqizli javob"** deb nomlanadi va kuchsiz giperbolik tasodifiy sistemalar uchun chiziqli jabov malalasiga bag'ishlanadi.

Tasodifiy dinamik sistema deb $(X,\mathcal{B})$ o'lchovli fazoda aniqlangan

$$S:\Omega\times X\to\Omega\times X,\ S(\omega,x)=(\sigma\omega,T_{\sigma(\omega)}(x)),$$

"qiya" ko'paytma akslantirishga aytiladi, bu yerda $\sigma:\Omega\to\Omega$ akslantirish $\mathbb{P}$ ehtimollik o'lchovini saqlaydi, T_ω akslantirishga esa "yaproq" akslantirishi deyiladi. Tasodifiy dinamik sistemalarni o'rganishda ikki xil yondoshuv mavjud: toblangan va so'ndirilgan. Toblangan holda dinamik sistemaning statsionar o'lchovlarining (bu o'lchovlar X fazoda aniqlanadi) xossalarini o'rganish nazarda tutiladi. Agar ixtiyoriy o'lchovli $A\subset X$ uchun

$$\int_X p(x,A)d\mu(x) = \mu(A),\ \ p(x,A) = \int_\Omega 1_A(T_\omega(x))d(\omega),$$

yoki, ixtiyoriy o‘lcholvi va chegaralangan $\phi: X \to \mathbb{R}$ uchun quyidagi

$$\int_X \int_\Omega \phi \circ T_\omega(x)d(\omega)d\mu(x) = \int \phi(x)d\mu(x).$$

tenglik bajarilsa, X fazoda aniqlangan μ o‘lchov statsionar o‘lchov deyiladi.

X kompakt interval va m undagi normallangan Lebeg o‘lchovi bo‘lsin. $T_\omega : X \to X$, $\omega \in \Omega$ akslantirishlar oilasi bo‘lib, har bir $\omega \in \Omega$ uchun Z_ω chekli yoki sanoqli to‘plam va $X_{z,\omega}$ ochiq intervallar X kompakt intervalning $z \in Z_\omega$ bo‘laklari bo‘lsin. Bundan tashqari, T_ω ning $X_{z,\omega}$ ga cheklanishi C^3 ga tegishli bo‘lib, ustiga akslantirish bo‘lsin. T_ω akslantirishning $X_{z,\omega}$ ga teskari tarmoqlarini $g_{z,\omega}$ bilan belgilaymiz. Barcha ω uchun bir xil belgilash to‘plami Z ni olamiz. Barcha akslantirishlar cheklangan yoki sanoqli tarmoqlarga ega bo‘lganligi sababli, z va ω mos kelmagan hollarda $g_{z,\omega}$ bo‘sh tarmoqlarni kiritish orqali muammo hal qilinadi. z bo‘yicha ko‘paytmalar paydo bo‘lganda, bo‘sh tarmoqlar hisobga olinmaydi. L_{T_ω} operatorni T_ω akslantirishning o‘tish operatori deb belgilaymiz, u quyidagicha aniqlanadi:

$$L_{T_\omega}\Phi = \sum_{z\in Z} \Phi \circ g_{z,\omega} \cdot | g_{z,\omega}'|.$$

Yuqoridagi tenglikni integrallab,

$$L\,\Phi := \int_\Omega L_{T_\omega}\Phi d(\omega).$$

tenglikni hosil qilamiz. Bu L – tasodifiy dinamik sistemaning o‘tish operatori deb ataladi. Aytaylik, Ω to‘plamda $\mathbb{P}_\varepsilon$ ehtimollik o‘lchovlari oilasi berilgan bo‘lsin. Biz ε soni 0 ning biror V atrofida $(\Omega,\{T_\omega\},\mathbb{P}_\epsilon)$ tasodifiy sistemaning statistik xossalarini o‘rganamiz. Qo‘zg‘atilgan sistemaning o‘tish operatori $L_{\mathbb{P}_\varepsilon}$ bilan belgilanadi. Aytaylik:

A-Faraz

(A1). $\tilde{D} > 0$ mavjud bo‘lib, barcha $x, y \in X$, $z \in Z$ va $\omega \in \Omega$ uchun quyidagi tengsizlik o‘rinli bo‘lsin:

$$\left| g_{z,\omega}'(x) / g_{z,\omega}'(y) - 1 \right| \le \tilde{D}\,|x - y|. \tag{16}$$

Shuningdek, ε ga bog‘liq bo‘lmagan $M > 0$ mavjud bo‘lib, $i = 2,3$ uchun quyidagi baho o‘rinli bo‘lsin:

$$\sup_{\varepsilon\in V}\sum_{z\in Z}\sup_{x\in X}\int_\Omega | g^{(i)}z,\omega | \, d(\omega) \le M. \tag{17}$$

(A2). shunday $\beta \in (0,1)$ topilib, $\sup_{\omega\in\Omega}\sup_{z\in Z}\sup_{x\in X} | g_{z,\omega}'(x) | \le \beta$ bo’ladi.

Agar T deterministik akslantirish qaralsa, (17) shart odatiy deformatsiya farazi $\sup_x | T_z'' x | / (T_z' x)^2 \le M$ tengsizlikdan kelib chiqadi. Bu yerda (17) shartda bunday deformatsiya "o‘rtacha hisobda" sodir bo‘lishini faraz qilyapmiz. Shuningdek (17) ifodada bo‘laklashlar bo‘yicha yig‘indi chekli bo‘lsin deb faraz qilyapmiz, chunki akslantirishlarning cheksiz bo‘laklarga ega bo‘lgan holni ham qaraymiz va Banach fazolarida aniqlangan o‘tish operatorlari bilan ishlaymiz, bu yerda $\|\cdot\|_{C^i}, i=0,1$ kuchsiz norma. Ayniqsa, (17) shart $\|L\ f\|_{C^i} \le C\|f\|_{C^i}, i=0,1$ tengizlikni qanoatlantiruvchi $\varepsilon>0$ o‘zgaruvchiga bog‘liq bo‘lmagan $C>0$ mavjudligini ta’minlash uchun talab qilinadi. Quyidagi tasdiq o‘rinli.

3-tasdiq. *(A1)-(A2) shartlar bajarilganda, har bir $\varepsilon\in V$ uchun L_ε operator C^1 va C^2 fazolarda ε bo‘yicha tekis spektral bo‘shliqqa ega. Qolaversa, $(\Omega,\{T_\omega\},P_\varepsilon)$ tasodifiy dinamik sistema yagona stasionar zichlikka ega va u $h_\varepsilon\in C^2$ shartni qanoatlantiradi.*

Har bir $z\in Z$ va $\Phi\in L^1(X)$ uchun quyidagicha belgilash kiritamiz:

$$\psi_z(\varepsilon,x)=\int_\Omega[\Phi\circ g_{z,\omega}\,|\,g_{z,\omega}'|](x)dP_\varepsilon(\omega).$$

B-faraz

- $L_{\mathbb{P}_\varepsilon}$ operator C^1 fazoda tekis spektral bo‘shliqqa ega bo‘lsin. Bundan tashqari, tasodifiy dinamik sistema $(\Omega,\{T_\omega\},P_\varepsilon)$ yagona $h_\varepsilon\in C^2$ stasionar zichlikga ega bo‘lsin.
- $\Phi=h_0\in C^2$ uchun $\partial_\varepsilon\psi_z(\varepsilon,x)$, $\partial_x\psi_z(\varepsilon,x)$, $\partial_x\partial_\varepsilon\psi_z(\varepsilon,x)$, $\partial_\varepsilon\partial_x\psi_z(\varepsilon,x)$ hosilalar mavjud va ular $X\times V$ da uzluksiz bo‘lsin. Shuning uchun, $\partial_\varepsilon\partial_x\psi_z(\varepsilon,x)=\partial_x\partial_\varepsilon\psi_z(,x)$. Bundan tashqari, $i=0,1$ uchun quyidagi shart o‘rinli bo‘lsin:

$$\sum_{z\in Z}\sup_{\varepsilon\in V}\sup_{x\in X}|\partial_x\psi_z^{(i)}(,x)|<\infty,$$

bu yerda $\psi_z^{(0)}=\psi_z$ va $\psi_z^{(1)}=\partial_x\psi_z$.

- Har qanday $\Phi\in C^1$ uchun $\psi_z(\varepsilon,x)$ va $\partial_x\psi_z(\varepsilon,x)$ mavjud bo‘lib, ixtiyoriy $(\varepsilon,x)\in X\times V$ va $i=0,1$ uchun quyidagi tengsizlik o‘rinli bo‘lsin:

$$\sum_{z\in Z}\sup_{\varepsilon\in V}\sup_{x\in X}|\psi_z^{(i)}(\varepsilon,x)|<\infty.$$

10-teorema. $(\Omega,\{T_\omega\},P_\varepsilon)$ *tasodifiy dinamik sistemalar oilasi yuqoridagi A va B farazlarni qanoatlantirsin. U holda, stasionar o‘lchovning zichligi $\varepsilon=0$ da differentsiallanuvchi, ya’ni $h^*\in C^1$ mavjud bo‘lib, quyidagicha ifodalanadi:*

$$\lim_{\varepsilon\to 0}\left\|\frac{h-h_0}{\varepsilon}-h^*\right\|_{C^1}\to 0. \tag{18}$$

Shuningdek, quyidagi chiziqli javob formulasi o‘rinli bo‘ladi

$$h^*:=(I-L_0)^{-1}\partial_\varepsilon L_\varepsilon h_0|_{\varepsilon=0}, \tag{19}$$

bu yerda

$$\partial_\varepsilon L_\varepsilon h_0|_{\varepsilon=0}=\partial_\varepsilon\sum_{z\in Z}\int_\Omega[h_0\circ g_{z,\omega}\,|\,g_{z,\omega}'|]dP_\varepsilon(\omega).$$

Aytaylik $\hat{\omega} \in \Omega^N$ berilgan bo‘lsin. $T_{\hat{\omega}}^n := T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} \circ T_{\omega_0} : X \to X$ belgilashlar kiritamiz: Ixtiyoriy $\hat{\omega} \in \Omega^N$ uchun $\Delta \subset \hat{\Omega}$ ga birinchi qaytish asklantirishini $\hat{T}_{\hat{\omega}}$ bilan belgilaymiz, ya’ni $x \in \Delta$ uchun

$$\hat{T}_{\hat{\omega}}(x) = T_{\hat{\omega}}^{R_{\hat{\omega}}(x)}(x), \ \ R_{\hat{\omega}}(x) = \inf\{n \geq 1 : T_{\hat{\omega}}^n(x) \in \Delta\}.$$

Z chekli, $z = z_0 z_1 \ldots z_n$ ko‘rinishidagi ketma-ketliklar to‘plami bo‘lsin, bu yerda $z_0 \in S^{in}$ va $z_i \in S^{out}$ barcha $i = 1, \ldots, n$ va $n \in \mathbb{N}$ uchun Δ interval ichida yotgan intervallarning indekslari to‘plami S^{in}, S^{out} esa aksincha uning tashqarisida yotdan intervallarning inekslari to‘plami. Ixtiyoriy $z \in Z$ uchun uning uzunligini $|z| = n+1$ bilan belgilaymiz. Shuningdek, $g_{z,\hat{\omega}} = g_{z_0,\omega_0} \circ g_{z_1,\omega_1} \circ \cdots \circ g_{z_n,\omega_n}$ belgilash ham ishlatamiz. U holda, $x \in X$ uchun $T_{\hat{\omega}}^{n+1} \circ g_{z,\hat{\omega}}(x) = x$ tenglik o‘rinli. Har bir $\hat{\omega}$ uchun $g_{z,\hat{\omega}}(\Delta)$ silindr to‘plamlar Δ ning (mod 0) bo‘laklashini hosil qiladi. $g_{z,\hat{\omega}}(\Delta)$ silindrda $R_{\hat{\omega}}(\cdot) = n+1$. Faraz qilaylik $(\Delta, \{\hat{T}_{\hat{\omega}}\}_{\hat{\omega} \in \Omega}, P^{\mathbb{N}})$ induksiyalangan akslantirish A va B farazlarni qanoatlantirsin hamda shoxlari C^3 ga tegishli va ustiga akslantirishlar bo‘lsin.

Induksiyalangan tasodifiy dinamik sistemalar. Quyidagicha belgilash kiritamiz $\hat{\Omega} = \Omega^{\mathbb{N}}$ va $\hat{\mathbb{P}} = \mathbb{P}^N$. Quyida $\hat{\omega}$ ning taqsimoti $\hat{\mathbb{P}}$ ga ko‘ra bog’liqsiz tanlangan $\hat{T}_{\hat{\omega}}$ akslantirishlarning kompozitsiyasidan iborat tasodifiy dinamik sistemani qaraladi. Yuqoridagi yondashuvga ko‘ra, ixtiyoriy $\Phi \in L^1(\Delta)$ funksiya uchun induksiya qilingan tasodifiy sistemaning o‘tish operatori quyidagi ko‘rinishga ega:

$$\hat{L}\,\Phi = \int_{\hat{\Omega}} \hat{L}_{\hat{\omega}} \Phi d(\hat{\omega}) = \sum_{z \in Z} \int_{\hat{\Omega}} \Phi \circ g_{z,\hat{\omega}} \,|\, g'_{z,\hat{\omega}} \,|\, d(\hat{\omega}), \tag{20}$$

bu yerda $\hat{T}_{\hat{\omega}}$ asklantirishning o‘tish operatori $\hat{L}_{T_{\hat{\omega}}}$ bilan belgilangan. $\hat{\mathbb{P}}$ ehtimollik o‘lchovining ko‘rinishini hisobga olsak, $\hat{L}$ operatorni quyidagicha yozish mumkin:

$$\hat{L}\,\Phi = \sum_{z \in Z} \int_{\Omega} \Phi \circ g_{z,\omega} \,|\, g'_{z,\omega} \,|\, d(\omega).$$

Induksiya qilingan tasodifiy sistemaning har qanday absolyut uzluksiz stasionar o‘lchovining $\hat{h}$ zichligi quyidagi tenglamani qanoatlantiradi:

$$\hat{L}\,\hat{h} = \hat{h}.$$

Induksiyalangan dinamik sistemani yoyish: Ixtiyoriy $\Phi \in L^1(\Delta)$, funksiya uchun $F_{\mathbb{P}}(\Phi) : X \to \mathbb{R}$ yoyish operatori quyidagicha aniqlanadi

$$F_{\mathbb{P}}(\Phi) := 1_{\Delta}\Phi + (1 - 1_{\Delta}) \sum_{z \in Z} \int_{\hat{\Omega}} \Phi \circ g_{z,\hat{\omega}} \,|\, g_{z,\hat{\omega}}' \,|\, d\hat{\mathbb{P}}(\hat{\omega}).$$

Ravshanki, $F_{\mathbb{P}}$ chiziqli operator.

4-Tasdiq. *Aytalik,* $\hat{h} \in L^1(\Delta)$ *va* $\hat{L}_{\hat{\mathbb{P}}}\hat{h} = \hat{h}$ *bo‘lsin. U holda* $L_{\mathbb{P}}(F_{\mathbb{P}}\hat{h}) = F_{\mathbb{P}}\hat{h}.$

Ushbu tasdiqdan $\hat{h}$ induksiyalangan sistemaning statsionar zichligi bo‘lsa, u holda $F_{\mathbb{P}}\hat{h}$ berilgan sistemaning statsionar zichligi bo‘lishi kelib chiqadi.

Qo‘zg‘atilgan tasodifiy sistema: Aytaylik, $\mathbb{P}_\varepsilon$ parametrlangan, Ω da aniqlangan, tashuvchisi $\bigcup_{k=1}^{\ell}\{k\}\times I_k$ bo‘lgan ehtimollik o‘lchovlari oilasi, V esa 0 ning biror atrofi bo‘lsin. Shuningdek, π_ε bilan $\mathbb{P}_\varepsilon$ o‘lchovning $\{1,\dots,\ell\}$ to‘plamdagi marginalini, $\eta_{k,\varepsilon}$ bilan esa $\mathbb{P}_\varepsilon$ ning $I_k\times\{k\}$ dagi shartli o‘lchovini belgilaylik. Yuqoridagi kabi, ε o‘zgarganda $(\Omega,\{T_\omega\},\mathbb{P}_\varepsilon)$ sistemaning statistik xossalari o‘zgarishini tadqiq qilamiz. Quyida $\hat{L}_{\hat{\mathbb{P}}_\varepsilon}$ operator C^1 fazoda tekis spektral bo‘shliqqa ega bo‘lsin, bundan tashqari, $(\hat{\Omega},\{\hat{T}_{\hat{\omega}}\},\hat{\mathbb{P}}_\varepsilon)$ tasodifiy dinamik sistemaning yagona statsionar zichligi $\hat{h}_\varepsilon\in C^2$ bo‘lsin deb faraz qilamiz. 4-tasdiqqa ko‘ra, berilgan $(\Omega,\{T_{\hat{\omega}}\},\mathbb{P}_\varepsilon)$ sistemaning statsionar zichligi va $(\hat{\Omega},\{\hat{T}_{\hat{\omega}}\},\hat{\mathbb{P}}_\varepsilon)$ induksiyalangan sistemaning statsionar zichligi[1] quyidagi formula orqali bog’langan

$$h_\varepsilon = F_{\mathbb{P}_\varepsilon}(\hat{h}_\varepsilon).$$

Berilgan dinamik sistemaning statsionar zichligini o‘zida saqlovchi Banach fazosi kiritish kerak. Umuman olganda, qaraladigan Banach fazosi berilgan sistema statsionar zichligining regulyarligiga bo‘g‘liq bo‘ladi.[2]

Aytaylik $\mathcal{H}$ fazo $(0,1]$ intervalda aniqlangan uzluksiz funksiyalar fazosi bo‘lib, unda

$$\| f\|_{\mathcal{H}} = \sup_{x\in(0,1]} |x^\gamma f(x)|,\ \gamma\ge 0,$$

norma kiritilgan bo‘lsin. Ixtiyoriy $z\in Z$ va $\Phi\in L^1(X)$ uchun quyidagi funksiyani aniqlaymiz:

$$\hat{\psi}_z(\Phi,x) = \int_{\hat{\Omega}}[\Phi\circ g_{z,\hat{\omega}} \,|\, g_{z,\hat{\omega}}'|](x)\,d\hat{\mathbb{P}}_\varepsilon(\hat{\omega}).$$

Bu funksiya quyidagi shartlarni qanoatlantirsin:

- Ixtiyoriy $\Phi=\hat{h}_0\in C^2$ uchun $\partial_\varepsilon\hat{\psi}_z(\varepsilon,x)$, $\partial_x\hat{\psi}_z(\varepsilon,x)$, $\partial_x\partial_\varepsilon\hat{\psi}_z(\varepsilon,x)$, $\partial_\varepsilon\partial_x\hat{\psi}_z(\varepsilon,x)$ xususiy hosilalar mavjud bo‘lib, $\Delta\times V$ da uzluksiz bo‘lsin. Shuning uchun, $\partial_\varepsilon\partial_x\hat{\psi}_z(\varepsilon,x)=\partial_x\partial_\varepsilon\hat{\psi}_z(\varepsilon,x)$ kommutativlik o‘rinli. Ixtiyoriy $\Phi\in C^1$ uchun $\hat{\psi}_z(\varepsilon,x)$ va $\partial_\varepsilon\hat{\psi}_z(\varepsilon,x)$ mavjud bo‘lib, $(0,1]\times V$ da uzluksiz. Qolaversa, $i=0,1$ uchun

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[1] Bu yerda h_ε normallanmagan. Agar h_ε integrallanuvchi bo’lsa, ya’ni tasodifiy sistemaning statsionar o’lchovi absolyute uzluksiz ehtimollik o’lchovi bo’lsa, u holda h_ε ning hosilasi topilgandan keyin normallangan zichlikning hosilasi osongina topiladi. Haqiqatan, agar $h_\varepsilon = h+\varepsilon h^* + o(\varepsilon)$, bo’lsa, u holda $\int h_\varepsilon = \int h + \int \varepsilon h^* + o(\varepsilon).$ Shuning uchun $\partial_\varepsilon(\frac{h_\varepsilon}{\int h_\varepsilon})|_{\varepsilon=0} = h^* - h\int h^*$.

[2] Masalan, agar statsionar o’lchov $x=1$ da maxsuslikka ega bo’lsa, u holda $\mathcal{H}$ ta’rifini shunga mos holda o’zgartirish mumkin. Yoki, agar statsinonar zichlik C^0 funksiya bo’lsa, u holda C^0 bilan ishlash mumkin.

$$\sum_{z\in Z}\sup_{\varepsilon\in V}\sup_{x\in\Delta}|\partial_\varepsilon\hat{\psi}_z^{(i)}(\varepsilon,x)|<\infty,$$

tengsizlik bajarilsin. Bu yerda $\hat{\psi}_z^{(0)}=\hat{\psi}_z$ and $\hat{\psi}_z^{(1)}=\partial_x\hat{\psi}_z$.

• Ixtiyoriy $\Phi\in C^1$ uchun $\psi_z(\varepsilon,x)$ va $\partial_x\psi_z(\varepsilon,x)$ mavjud va uzluksiz bo‘lsin. Bundan tashqari $i=0,1$ uchun quyidagi tengsizlik bajarilsin:

$$\sum_{z\in Z}\sup_{\varepsilon\in V}\sup_{x\in\Delta}|\hat{\psi}_z^{(i)}(\varepsilon,x)|<\infty.$$

• Ixtiyoriy $\Phi\in C^0$ uchun quyidagi tengsizlik bajarilsin:

$$\sum_{z\in Z}\sup_{\varepsilon\in V}\|\hat{\psi}_z(\varepsilon,\cdot)\|_{\mathcal{H}}<\infty.$$

• Ixtiyoriy $\Phi=\hat{h}\in C^2$ uchun quyidagi tengsizlik bajarilsin

$$\sum_{z\in Z}\sup_{\varepsilon\in V}\|\partial_\varepsilon\hat{\psi}_z(\varepsilon,\cdot)\|_{\mathcal{H}}<\infty.$$

11-teorema. *Aytaylik* $(\Omega,\{T_\omega\},\mathbb{P}_\varepsilon)$ *yuqoridagi shartlarni qanoatlantiruvchi tasodifiy dinamik sistemalar oilasi bo‘lsin. U holda,*

(1) *shunday* $h^*\in\mathcal{H}$ *topilib*

$$\lim_{\varepsilon\to 0}\|\frac{h_\varepsilon-h_0}{\varepsilon}-h^*\|_{\mathcal{H}}=0;$$

tenglik o‘rinli bo‘ladi, ya’ni h_ε *zichlik* $\mathcal{H}$ *Banach fazosida* ε *bo‘yicha* $\varepsilon=0$ *da differensialanuvchi.*

(2) *Agar,* $\mathcal{H}$ *ning normasi ta’rifida* $\gamma<1$ *bo‘lsa,*

$$\lim_{\varepsilon\to 0}\|\frac{h_\varepsilon-h_0}{\varepsilon}-h^*\|_1=0.$$

Dissertatsiyaning to‘rtinchi bobi **"Kuchsiz giperbolik tasodifiy dinamik sistemalar"** deb nomlanadi va tasodifiy dinamik sistemalarni so‘ndirilgan nuqtai nazardan o‘rganamiz, agar tasodifiy sistema o‘lchovni saqlovchi (Ω,σ,P) sistema ustida $f_\omega: X\to X$ yaproq akslantirishlar oilasi bilan berilgan bo‘lsa, quyidagi kosikllaning P – dinamikasini deyarli barcha ω uchun o‘rganish talab etiladi:

$$f_\omega^0=Id,\ f_\omega^n=f_{\sigma^{n-1}\omega}\circ f_{\sigma^{n-2}\omega}\circ\ldots\circ f_\omega$$

Agar $(f_\omega)_*\mu_\omega=\mu_{\sigma\omega}$ bo‘lsa, u holda o‘lchovlar oilasi $\{\mu_\omega\}$ equivariant deyiladi.

Kelgusi va *o‘tgan so‘ndirilgan korrelatsiyalar*: Berilgan $\varphi,\psi: X\times\Omega\to\mathbb{R}$ uchun *kelgusi* va *o‘tgan so‘ndirilgan* korrelatsiyalar quyidagicha aniqlanadi:

$$Cor_{n,\omega}^{(f)}(\varphi,\psi)=\int(\varphi_{\sigma^n\omega}\circ f_\omega^n)\psi_\omega d\mu_\omega-\int\varphi_{\sigma^n\omega}d\mu_{\sigma^n\omega}\int\psi_\omega d\mu_\omega,$$

$$Cor_{n,\omega}^{(p)}(\varphi,\psi)=\int(\varphi_\omega\circ f_{\sigma^{-n}\omega}^n)\psi_{\sigma^{-n}\omega}d\mu_{\sigma^{-n}\omega}-\int\varphi_\omega d\mu_\omega\int\psi_{\sigma^{-n}\omega}d\mu_{\sigma^{-n}\omega}.$$

6-ta’rif. $\Omega\times X$ *to‘plamda ikkita* $\mathcal{B}_1$ *va* $\mathcal{B}_2$ *Banach fazo berilgan bo‘lsin. Musbat, kamayuvchi* $\{\rho_n\}_{n\in\mathbb{N}}$ *sonlar ketma-ketligi uchun* $\lim_{n\to\infty}\rho_n=0$ *bo‘lsin.* P *-deyarli barcha* ω, *ixtiyoriy* $\varphi\in\mathcal{B}_1$, $\psi\in\mathcal{B}_2$ *funksiyalar uchun shunday* C_ω *va* $C_{\varphi,\psi}$ *mavjud bo‘lib,*

$$|Cor_{n,\omega}^{(f)}(\varphi,\psi)|\le C_\omega C_{\varphi,\psi}\rho_n, \quad |Cor_{n,\omega}^{(p)}(\varphi,\psi)|\le C_\omega C_{\varphi,\psi}\rho_n$$

tengsizliklar o'rinli bo'lsa, f_ω akslantirish uchun ρ_n so'ndirilgan korrelyatsiyalar kamayish tezligi deyiladi.

3-Izoh. *Agar C_ω tasodifiy o'zgaruvchi P-integralanuvchi bo'lsa, u holda so'ndirilgan korrelyatsiyalarni integrallab, $\int_\Omega Cor_{n,\omega}^{(f)}(\varphi,\psi)dP\le \hat{C}_{\varphi,\psi}\rho_n$ baho olish mumkin. Bu holda, toblangan korrelyatsiyalar bilan quyidagicha bog'liqlik hosil bo'ladi*

$$Cor_n(T,\varphi,\psi)=\int_\Omega Cor_{n,\omega}^{(f)}(\varphi,\psi)dP+Cor_n(\sigma,\bar{\varphi},\bar{\psi}),$$

bu yerda : $\bar{\varphi}_\omega:=\int_X\varphi d\mu_\omega$ va $\bar{\psi}_\omega:=\int_X\psi d\mu_\omega$.

Quyida so'ndirilgan korrelyatsiyalarning kamayish tezligini baholash uchun abstrakt sxema taklif qilinadi. Bu sxema kuchsiz giperbolik sistemalar uchun yaroqli ekani misollarda tushuntiriladi.

(A,p) ehtimollik fazosida $\Omega=A^{\mathbb{Z}}$, $P=p^{\mathbb{Z}}$ belgilash kiritib, $\sigma:\Omega\to\Omega$ surish akslantirishini aniqlaymiz. (Ω,σ,P) sistema ustida $f_\omega:X\to X$ yaproq aklantirishlar oilasi berilgan bo'lib, $f_\omega=f_{\omega_0}$ tenglik o'rinli bo'lsin. Shuningdek, $\Lambda\subset X$ to'plamning o'lchovi uchun $m(\Lambda)=1$ bo'lsin. Agar P o'lchov bo'yicha deyarli barcha $\omega\in\Omega$ uchun Λ to'plamning $\{\Lambda_j(\omega)\}_j$ sanoqli bo'laklashlari va $R_\omega:\Lambda\to\mathbb{N}$ o'lchovli qaytish vaqti mavjud bo'lib u har bir $\Lambda_j(\omega)$ bo'lakda o'zgarmas bo'lsa, hamda $f_\omega^{R_\omega}(x)=f_{\sigma^{R_\omega(x)-1}\omega}\circ\cdots\circ f_{\sigma\omega}\circ f_\omega(x)\in\Lambda$ shart P o'lchov bo'yicha deyarli barcha $\omega\in\Omega$ va m o'lchov bo'yicha deyarli barcha $x\in\Lambda$ uchun o'rinli bo'lsa $\{f_\omega\}_{\omega\in\Omega}$ tasodifiy akslantirish $\Lambda\subset X$ to'plamda tasodifiy minoraga ega deyiladi. Yuqoridagi qaytish vaqtiga ko'ra tasodifiy minora deyarli barcha ω lar uchun quyidagicha aniqlanadi

$$\Delta_\omega=\left\{(x,\ell)\in\Lambda\times\mathbb{Z}_+\mid x\in\cup_j\Lambda_j(\sigma^{-\ell}\omega),j,\ell\in\mathbb{Z}_+,0\le\ell\le R_{\sigma^{-\ell}\omega}(x)-1\right\}$$

va minora akslantirishi $F_\omega:\Delta_\omega\to\Delta_{\sigma\omega}$

$$F_\omega(x,\ell)=\begin{cases}(x,\ell+1), & agar \quad \ell+1<R_{\sigma^{-\ell}\omega}(x)\\ (f_{\sigma^{-\ell}\omega}^{\ell+1}x,0), & agar \quad \ell+1=R_{\sigma^{-\ell}\omega}(x),\end{cases}$$

ko'rinishda yoziladi. Minora quyidagi shartlarni qanoatlantirsin deb faraz qilamiz:

(P1) **Markov**: har bir $\Lambda_j(\omega)$ uchun $F_\omega^{R_\omega}|\Lambda_j(\omega):\Lambda_j(\omega)\to\Lambda$ akslantirish biyeksiya.

(P2) **Cheklangan deformatsiya**: $D>0$ va $0<\gamma<1$ o'zgarmaslar mavjud bo'lib, barcha ω va har bir $\Lambda_j(\omega)$ uchun $F_\omega^{R_\omega}|\Lambda_j(\omega)$ akslantirish va uning teskarisi m bo'yicha nosingulyar bo'ladi, va $JF_\omega^{R_\omega}|\Lambda_j(\omega)$ Jakobi determinanti musbat bo'lib, har bir $x,y\in\Lambda_j(\omega)$ uchun quyidagi tengsizlikni qanoatlantiradi

$$\left|\frac{JF_\omega^{R_\omega}(x)}{JF_\omega^{R_\omega}(y)}-1\right|\le D\gamma^{s(F_\omega^{R_\omega}(x,0),F_\omega^{R_\omega}(x,0))}.$$

(P3) **Kuchsiz kengaytirish**: $\mathcal{P}_\omega$ bo‘laklash F_ω uchun hosil qiluvchi bo‘laklashdir, ya’ni $\vee_{j=0}^{n}F_\omega^{-j}\mathcal{P}_{\sigma^j\omega}$ bo‘laklahlarning diametrlari n cheksizga yaqinlashganda nolga yaqinlashadi;

(P4) A**simptotik qaytish vaqti**: $C>0$, $a>1$, $b\ge 0$, $u>0$, $v>0$ o‘zgarmaslari, to‘liq o‘lchovli $\Omega_1\subset\Omega$ qism to‘plam va $n_1:\Omega_1\to\mathbb{N}$ tasodifiy o‘zgaruvchi mavjud bo‘lib, quyidagi tengsizlik o‘rinli

$$\begin{cases} m\{x\in\Lambda\,|\,R_\omega(x)>n\}\le C\dfrac{(\log n)^b}{n^a}, \text{agar}\, n\ge n_1(\omega),\\ P\{n_1(\omega)>n\}\le Ce^{-un^v}; \end{cases}$$

(P5) **Aperiodiklik**: $N\in\mathbb{N}$ va $\{t_i\in\mathbb{Z}_+\,|\,i=1,2,...,N\}$ mavjud bo‘lib, EKUB $\{t_i\}=1$ va $\epsilon_i>0$ sonlari topilib, deyarli har bir $\omega\in\Omega$ va $i=1,2,\dots N$ uchun $m\{x\in\Lambda\,|\,R_\omega(x)=t_i\}>\epsilon_i$ tengsizlik bajariladi.

(P6) **Chegaralanganlik**: $M>0$ mavjud bo‘lib, $\int m(\Delta_\omega)dP(\omega)\le M$.

(P7) A**simptotik o‘rtacha qaytish vaqti**: $C>0$, $\hat{b}\ge 0$ va $a>1$ o‘zgarmaslar mavjud bo‘lib $\int_\Omega m\{x\in\Lambda\,|R_\omega=n\}dP\le C\dfrac{(\log n)^{\hat{b}}}{n^{a+1}}$ tengsizlik o‘rinli bo‘ladi.

Δ to‘plamda funksional fazolarni aniqlaymiz. Quyida $u>0$, $v>0$, $a>1$, $b\ge 0$, $0<\gamma<1$ o‘zgarmaslarni (P2) va (P4) shartlarga muvofiq tanlaymiz

$$F_\gamma^+=\{\varphi_\omega:\to\mathbb{R}\,|\quad \exists C_\varphi>0,\forall I_\omega\in_\omega, \text{yoki } \varphi_\omega\,|\,I_\omega\equiv 0$$
$$\text{yoki}\,\varphi_\omega\,|\,I_\omega>0\,\text{va}\left|\log\frac{\varphi_\omega(x)}{\varphi_\omega(y)}\right|\le C_\varphi\gamma^{s(x,y)},\forall x,y\in I_\omega\}.$$

$K_\omega:\Omega\to\mathbb{R}_+$ tasodifiy miqdor $\inf_{\omega\in\Omega}K_\omega>0$ va

$$P\{\omega\,|\,K_\omega>n\}\le e^{-un^v}. \tag{42}$$

shartlarni qanoatlantiradi. Tasodifiy, chegaralangan funksiyalar

$$L_\omega^{K_\omega}=\{\varphi_\omega:\Delta\to\mathbb{R}\,|\,\exists C_\varphi^{'}>0,\sup_{x\in\Delta_\omega}|\,\varphi_\omega(x)\,|\le C_{\varphi'}K_\omega\}$$

fazosiga mos tasodifiy Lipshitz funksiyalar fazosi quyidagicha kiritiladi

$$L_\omega^\infty=\{\varphi_\omega\in L_\omega^{K_\omega}\,|\,\exists C_\varphi>0,|\,\varphi_\omega(x)-\varphi_\omega(y)\,|\le C_\varphi K_\omega\gamma^{s(x,y)},\forall x,y\in I_\omega\}.$$

7-ta’rif. (i) Agar $\vee_{n=0}^{\infty}F^{-n}\mathcal{B}$ bo‘laklash trivial bo‘lsa $(F,\nu)=(F_\omega,\nu_\omega)_{\omega\in\Omega}$ tasodifiy sistema aniq deyiladi.

Trivial bo‘laklash degani ixtiyoriy $B\in\vee_{n=0}^{\infty}F^{-n}\mathcal{B}$ va deyarli barcha ω uchun, $\nu_\omega(B)=0$ yoki $\nu_\omega(B)=1$ tengliklardan faqat bittasi bajarilishini anglatadi.

(ii) (F,ν) tasodifiy sistema aralashtiruvchi deyiladi agar barcha $\varphi,\psi \in L^2(\nu)$ uchun quyidagi tenglik to‘g’ri bo‘lsa:

$$\lim_{n\to\infty}\left|\int_\Omega\int\varphi_{\sigma^n\omega}\circ F_\omega^n\cdot\psi_\omega d\nu_\omega dP-\int_\Omega\int\varphi_\omega d\nu_\omega dP\int_\Omega\int\psi_\omega d\nu_\omega dP\right|=0.$$

12-teorema. *Deyarli barcha $\omega\in\Omega$ uchun, m ga nisbatan ekvivalent va absolyut uzluksiz $\nu_{\sigma^k\omega}$ oila mavjud va bu oila deyarli barcha $\omega\in\Omega$ uchun $(F_{\sigma^k\omega})_*\nu_{\sigma^k\omega}=\nu_{\sigma^{k+1}\omega}$, va $\nu_\omega=h_\omega m$ tengliklarni qanoatlantiradi. Shuningdek, deyarli har bir $\omega\in\Omega$ uchun $h_\omega\in F_\gamma^{K_\omega}\cap F_\gamma^+$ shartni qanoatlantiradigan K_ω tasodifiy miqdor mavjud bo‘ladi.*

13-teorema. *Aytaylik $\delta>0$ va 12-teoremadagi K_ω tasodifiy miqdor berilgan bo‘lsin. Ixtiyoriy $\varphi\in L_\infty^{K_\omega}$, $\psi\in L_\gamma^{K_\omega}$ va deyarli barcha $\omega\in\Omega$ uchun C_ω tasodifiy o‘zgaruvchi va o‘zgarmas $C_{\varphi,\psi}$ mavjud bo‘libular uchun quyidagilar o‘rinli.*

- *“Kelajak" operatsion korrelyatsiyalari:*

$$\left|\int(\varphi_{\sigma^n\omega}\circ F_\omega^n)\psi_\omega dm-\int\varphi_{\sigma^n\omega}d\nu_{\sigma^n\omega}\int\psi_\omega dm\right|\le C_\omega C_{\varphi,\psi}n^{1+\delta-a}.$$

- *“O‘tmish" operatsion korrelyatsiyalari:*

$$\left|\int(\varphi_\omega\circ F_{\sigma^{-n}\omega}^n)\psi_{\sigma^{-n}\omega}dm-\int\varphi_\omega d\nu_\omega\int\psi_{\sigma^{-n}\omega}dm\right|\le C_\omega C_{\varphi,\psi}n^{1+\delta-a}.$$

Shuningdek, $C>0$, $u'>0$, $v'\in(0,1)$ o‘zgarmaslar mavjud bo‘lib, $P\{C_\omega>n\}\le Ce^{-u'n^{v'}}$ tengsizlik o‘rinli bo‘ladi.

XULOSA

Mazkur dissertatsiya dinamik sistemalarda boshqaruvchanlik, asimptotik turg‘unlik, tasodifiy dinamik sistemalarda statistik turg‘unlik, chiziqli javob va korrelyatsiyalarning kamayishi, sistemani tavsiflovchi o‘lchovning asimptotik turg‘unligi kabi masalalarni tadqiq etishga bag‘ishlangan. Dissertatsiyaning asosiy natijalari quyidagilardan iborat:

Banach fazosidagi sistemaning turg‘unligi normaga bog‘liq ekanligi ko‘rsatildi. Katta spektrli oddiy misolda cheksiz o‘lchovli fazolarda yangi hodisalar yuz berishi mumkinligi aniqlandi. Agar tizim spektri diskret bo‘lgan operatorlar bilan “yaxshi” yaqinlashtirilsa, unda turg‘unlik va boshqaruvchanlik holati chekli o‘lchovli holatga o‘xshashligi ko‘rsatildi.

Lorents oqimi misolida, agar sistema va uning buzilishlari ma’lum bir giperboliklikni saqlab qolsa, statistik turg‘unlikni isbotlash mumkinligi ko‘rsatildi.

Shuningdek, agar sistema yetarlicha giperbolik bo‘lib parametr bo‘yicha silliq bo‘lsa, u holda chiziqli javob mavjud bo‘lishi isbotlandi.

Kuchsiz giperbolik tasodifiy sistemalarda korrelyatsiyalarning kamayishini isbotlash uchun abstrakt usul ishlab chiqildi. Ushbu usullar deterministik analoglari kabi samaralidir va kuchsiz giperbolik tasodifiy dinamik sistemalarning prototipik misollariga qo‘llanilishi mumkin.

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INSTITUTE OF MATHEMATICS

RUZIBOEV MARKS BOTIRBOEVICH

ERGODICITY, STABILITY AND CONTROL OF DYNAMICAL SYSTEMS

01.01.02-Diffrential Equations and Mathematical Physics

**ABSTRACT OF DISSERTATION OF THE DOCTOR OF SCIENCE (DSc)
ON PHYSICAL AND MATHEMATICAL SCIENCES**

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Scientific consultant:	**Azamov Abdulla Azamovich** Doctor of Physical and Mathematical Sciences, academician
Official opponents:	**Akhtam Dzhalilov Abdurakhmanovich** Doctor of Physical and Mathematical Sciences, professor
	Durdiyev Durdimurad Kalandarovich Doctor of Physical and Mathematical Sciences, professor
	Sachkov Yuriy Leonidovich Doctor of Physical and Mathematical Sciences, professor
Leading organization:	**The National University of Uzbekistan**

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Dissertation is possible to review in Information-resource centre at Institute of Mathematics named after V.I.Romanovsky (is registered № 192) (Address: University str. 9, Almazar area, Tashkent, 100174, Uzbekistan, Ph.: (+99878) 207-91-40).

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O'ZBEKISTON RESPUBLIKASI FANLAR AKADEMIYASI V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI

U.A. Rozikov
Chairman of Scientific Council on award of scientific degrees, d.f.-m.s., Academician

J.K. Adashev
Scientific secretary of Scientific Council on award of scientific degrees, d.f.-m.s., Senior researcher

R.R. Ashurov
Deputy Chairman of scientific Seminar under Scientific Council on award of scientific degrees,d.f.-m.s., Professor

INTRODUCTION (abstract of DSc thesis)

Actuality and demand of the theme of the dissertation. A significant part of global scientific and practical research possesses an evolutionary nature, changing according to specific laws over time. Such changes are referred to as dynamical processes, which are modeled using differential equations. Studying the behavior of these processes is effectively done through the theory of dynamical systems. Mathematical modeling of dynamic processes involves identifying the laws governing the system and analyzing the parameters that characterize it. In modeling natural processes, approximate calculations of parameter values are often necessary. This raises the question of how the dynamics of mathematical models change under slight variations in parameter values, known as the stability problem of dynamical systems, which is of great practical importance. Studying the stability of a dynamic system under external influences and describing its evolution is one of the pressing issues. The concept of “stability” can be defined in various ways, leading to different branches of dynamical systems theory.

Currently, the theory of dynamical systems and its applications are being extensively studied worldwide. The theory was founded by A. Poincaré in the late 19th century and has since become a classical field of mathematics. Despite this, due to the diversity of problems and their applications across various fields, it remains one of the most relevant and rapidly developing areas of modern mathematics. This is evidenced by the fact that, over the past thirty years, seven recipients of the Fields Medal, the highest honor in mathematics, have been specialists in the theory of dynamical systems.

In our country, significant achievements have been made in the stability theory of dynamical and controlled systems, thanks to increased attention[1] to fundamental research aimed at gaining a leading position globally. Strengthening research in dynamical systems theory, specifically studying the stability of systems concerning parameters, is of crucial importance.

The research in this dissertation serves to implement the tasks outlined in the Decree of the President of the Republic of Uzbekistan No. PF-4947 dated February 7, 2017, "On the Strategy for Further Development of the Republic of Uzbekistan," the Resolution No. PQ-4387 dated July 9, 2019, "On State Support for the Further Development of Mathematics Education and Sciences, as well as Fundamental Measures to Improve the Activities of the Institute of Mathematics Named After V.I. Romanovskiy of the Academy of Sciences of the Republic of Uzbekistan," and the Resolution No. PQ-4708 dated May 7, 2020, "On Measures to Improve the Quality of Education and Develop Research in the Field of Mathematics," along with other regulatory-legal documents related to this activity.

Alignment with the priority areas of national science and technology development. This research corresponds to the fourth priority direction, "Mathemat-

[1] Resolution No. PQ-4387 of July 9, 2019, by the President of the Republic of Uzbekistan "On measures for further state support of mathematics education and sciences, as well as the comprehensive improvement of the activities of the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan."

ics, Mechanics, and Informatics," of the development of national science and technology.

Review of foreign scientific research on the dissertation topic. Scientific research on dynamical systems is widely conducted in leading universities such as Princeton, Massachusetts Institute of Technology, Maryland, Pennsylvania State University, Georgia Institute of Technology (USA), Sorbonne University (France), University of Zurich (Switzerland), Warwick University (UK), the Steklov Institute of Mathematics of the Russian Academy of Sciences, Kyushu and Keio Universities (Japan), and the Southern University of Science and Technology (China), covering various aspects of the field.

In recent years, significant scientific results have been obtained in the statistical study of dynamical systems. For instance, the exponential decay rate of the correlation function for the geometric Lorenz attractor has been proven (Warwick University, UK), and similar findings for the "Sinai billiard" system's flow (collaboration between researchers from Italy’s Tor Vergata, France’s Sorbonne, and New York University) indicate exponential convergence to equilibrium measures. Additionally, invariant measures for smooth Anosov diffeomorphisms have been proven to be stable and even smooth under external influences (Institute des Hautes Études Scientifiques, France).

The theory of stochastic dynamical systems, the counterpart of stochastic differential equations, has also been rapidly developing, with research conducted in Australia, Italy, France, Croatia, the UK, and the USA.

The degree of scrutiny of the problem. The theory of stability in dynamic systems is associated with the names of A. Poincaré, A. M. Lyapunov, A. Andronov, and L. S. Pontryagin. Stability in the sense of Lyapunov refers to the topological stability of the system's solution. Since stability is a mathematical expression of steady motion, it found applications in many problems in mechanics. Later, in 1937, Pontryagin and Andronov introduced the concept of structural stability of dynamic system trajectories and proved necessary and sufficient conditions for the stability of structures in plane systems. In the 1960s, M. Peixoto and M.C. Peixoto found sufficient conditions for the structural stability of dynamic systems and showed that systems satisfying these conditions are dense in the family of diffeomorphisms. During the same period, D. Anosov proved the structural stability of hyperbolic automorphisms of the torus and generalized this class, which became known as Anosov diffeomorphisms. S. Smale further generalized this concept, introducing a class called "Axiom A". Smale and his school demonstrated that the generalization of Andronov-Pontryagin systems in higher-dimensional spaces is known today as Morse-Smale systems.

Alongside the topological properties of dynamic systems, the study of hyperbolic dynamic systems from a statistical perspective began. E. Hopf and D. Anosov showed that hyperbolic systems are ergodic, while the introduction of Markov partitions for torus automorphisms by Ya. Sinai enabled the application of statistical physics methods to the study of dynamics. Later, R. Bowen constructed similar partitions for "Axiom A" systems. These systems were shown to have special Sinai-Ruelle-Bowen measures, exponential decay of the correlation function, the central limit theorem, and several other statistical properties. The connection between hyperbolic systems and geometry, as well as spectral theory, brought many new ideas to the field. For example, geodesic flows on manifolds with negative Gaussian curvature are Anosov (hyperbolic) flows. These flows are ergodic, their correlation coefficients decrease exponentially, and the eigenvalues of the Laplace-Beltrami operator defined on the manifold are related to the eigenvalues of the transfer operator defined by the flow. A major limitation of hyperbolic systems is that they cannot be defined on just any manifold. Therefore, in the last 30 years, there has been active research to identify a broader class that includes hyperbolic systems and can be defined on any manifold as a phase space.

Currently, there are numerous research schools in our country and abroad in various directions of this field. The theory of stability and its direct continuation in optimal control theory and differential games have seen significant contributions from our country's scientists in collaboration with foreign colleagues. N. Satimov developed the so-called 3rd pursuit method. Additionally, A. Azamov contributed to applying generalized characteristic indices in the stability theory of distributed parameter systems, while results were obtained on the properties of the alternating integral proposed by Pontryagin. The theory of differential games further developed, yielding significant outcomes for systems represented by infinite-dimensional systems, systems defined on Riemannian manifolds, and systems representing the motion of inertial objects and those with delayed parameters, as achieved by M. To'xtasinov, G. Ibragimov, O. Qo'chqorov, B. Samatov, and N. Mamadaliyev.

The relevance of the dissertation research to the research plans of the scientific research institution where the dissertation was conducted. The dissertation research is directly related to the scientific research work carried out under the research plans of the Institute of Mathematics named after V.I. Romanovskiy within the framework of the fundamental project OT-F4-84 "Discrete-numerical method for polynomial systems and its applications to modeling cyclic and controlled processes" (2017-2020) and the research programs of the project "Mathematical models of dynamic processes and their applications" (2020-2024).

The aim of the research consists of proving statistical stability, contiutiy and smoothness of the invariant measure with respect to the external parameter, asymptotic stability of invariant measures and controllability problems.

The objectives of the research include:

studying the asymptotic stability and control problems in controllable systems defined in infinite-dimensional spaces;

investigating pursuit-evasion problems in differential games defined in finite-dimensional spaces where integral constraints are imposed on the players’ controls;

proving the statistical stability of the Lorenz attractor;

developing an abstract method to prove the smoothness of stationary measures in random dynamical systems and applying it to Gauss-Renyi and Manneville-Pomeau maps;

Constructing a random analog of the abstract Young Tower to estimate the decay rate of the correlation function for almost all realizations in random dynamical systems and applying it to weakly hyperbolic systems.

Object of the research: Linear control systems and weakly hyperbolic chaotic dynamical systems.

Subject of the research: Stability of dynamic and controllable systems.

Research methods: The dissertation applies methods from differential equations, operator theory, dynamical systems theory, and probability theory.

Scientific novelty of the research includes:

a new class of controllable systems defined in infinite-dimensional spaces has been proven to be asymptotically stable and null-controllable;

the evasion problem has been solved for differential games in finite-dimensional spaces, with integral constraints on the controls of the players;

the statistical stability of the Lorenz attractor and the stability of the variance for this system have been proven;

an abstract method has been developed to prove the smoothness of stationary measures in random dynamical systems, using this abstract method, the smoothness of the invariant measures has been proven for the Gauss-Reiny and Manneville-Pomeau mappings;

a random analog of the abstract Young Tower has been developed to estimate the rate of decay of the correlation function in almost all realizations of random dynamical systems, this method was applied to systems with weak hyperbolicity and their perturbations to obtain estimates for the rate of decay of the correlation functio;

the existence of a critical intermittency phenomenon has been proven for a system formed by random compositions of mappings belonging to the logistic family.

Reliability of research results. The reliability of the results is supported by rigorous mathematical proofs based on the methods of differential equations, functional analysis, probability theory, and the theory of dynamical systems.

Scientific and practical significance of the research results. The scientific significance of the research results lies in the development of methods for the statistical study of weakly hyperbolic random dynamical systems. The practical significance is explained by the application of these results to the qualitative theory of dynamical systems.

Implementation of research results. Based on the results obtained in the study of stability, ergodicity, and control problems in dynamic systems:

the abstract method developed to prove the smoothness of stationary measures in random dynamical systems was used to solve the linear response problem for random dynamical systems in the international project numbered IP-2019-04-1239 titled "Transfer operators, limit laws, and infinite-dimensional dynamical systems". (Confirmed by a reference from the University of Rijeka, Croatia, dated November 21, 2023.) The application of the scientific results made it possible to solve the linear response problem in random dynamical systems with weak hyperbolicity.

The statistical stability of Lorenz attractors and the results obtained for controllable systems defined in infinite-dimensional spaces were used in the project numbered OT-F4-(36+32) titled “Development of new methods for solving mathematical physics and optimal control problems. Nonclassical initial and boundary value problems for equations with odd-order partial derivatives and their applications” to solve the nonlinear problems arising in the developing the new methods for solving optimal control problems and their numerical implementation (Confirmed by a reference letter of the National University of Uzbekistan, dated December 28, 2023, No. 04/11-9455.). The application of the scientific results allowed the construction of optimal control synthesis for systems described by infinite differential equations.

An abstract method developed for estimating the decay rate of the correlation function in systems with weak hyperbolicity was utilized in scientific articles published in international journals to estimate the decay rate of the correlation function (Advances in Math, 2023, vol. 426; Ann. Probab., 2022, 50(1), 241–303; Commun. Math. Phys., 2021, 385, 905–935). The application of these scientific results allowed the derivation of estimates for the decay of the correlation function in systems with weak hyperbolicity.

Approbation of research results. These research results were discussed at eight international scientific-practical conferences held in leading scientific centers worldwide.

Publication of research results. A total of 25 scientific papers have been published based on the dissertation results, including 13 articles in scientific publications recommended by the Higher Attestation Commission of the Republic of Uzbekistan for publishing the main scientific results of doctoral dissertations (DSc), and 12 articles in journals indexed in the “Scopus” and “Web of Science Core Collection” databases.

The structure and volume of the dissertation: The dissertation consists of an introduction, four chapters, and a list of references, totaling 197 pages.

THE MAIN CONTENT OF THE DISSERTATION

In the introduction the motivation of research theme and correspondence to the priority research areas of science and technology of the Republic, we present a review of international research on the theme of the dissertation and the degree of scrutiny of the problem, formulate our goals and objectives, identify the object and subject of study, and state scientific novelty and practical results of the research. Moreover, we discuss the theoretical and practical applications of the results, and provide information on the implementation of the research results, the list of publications and the structure of dissertation.

The first chapter of the dissertation is titled **“Controllable dynamical systems with and without conflicted parties”** and it contains the results in the direction of control theory and asymptotic stability. We consider a controlled system with distributed parameters, described by evolution-type linear partial differential equations

$$w_t = Aw + v, \tag{1}$$

where $w(x,t)-$ is a scalar function of the vector $x=(x_1,...,x_d)$ and time $t,$ which describes the state of the system, v is a control parameter, also chosen in the form of the function $v(t,x),$ A is a linear elliptic differential operator of order $2m$.

Equation (1) is given in the region $\Delta = \bar{\Omega} \times [0,+\infty),$ where Ω is an open domain in R^d with a compact closure $\bar{\Omega}$ and regular boundary Γ and is considered for the following initial and boundary conditions:

$$w(x,0) = w_0(x),\ x \in \Omega, Mw = 0,\ x \in \Gamma;\ M = (M_1, M_2, ..., M_m). \tag{2}$$

where $M_1, M_2 ..., M_m$ are linear differential operators whose order is strictly less than $2m$.

The control function v is chosen from the class $L_2(\Delta)$ and must satisfy the constraint

$$|v(x,t)| \le v^0,\ x \in \Omega,\ t \ge 0, \tag{3}$$

where v^0 is a fixed positive constant. These functions will be called admissible controls.

The control problem is formulated as follows: it is required to obtain the admissible control $v(x,t),$ such that the solution $w(x,t)$ corresponding to it from the class $W^{2m,1}(\Delta)$ of equation (1) with conditions (2) becomes identically zero at a certain instant of time $T, T>0,$ i.e. $w(x,T)=0$ for almost all $x \in \Omega$. Each control, which possesses this property, will be called a guaranteeing control. Henceforth we will call the quantity $T,$ for which the condition $w(x,T) \equiv 0$ is satisfied, the transition time. The quantity $T^* = \inf T,$ where the infimum is taken over all possible guaranteeing controls, will be called the optimal transition time.

We assume that the operator A possesses a complete system of eigenfunctions $\{\varphi_k(x)\}$ and the solution $w(x,t)$ and the control $v(x,t)$ from the classes considered allow of expansions in this system:

$$w(x,t) = \sum_k q_k(t)\varphi_k(x), \quad v(x,t) = \sum_k \tilde{u}_k(t)\varphi_k(x)$$

Thus, the pair of functions $w(x,t)$, $v(x,t)$ satisfies relations (1) – (3) if and only if the quantities q_k and $\tilde{u}_k$ satisfy the infinite system of equations

$$\dot{q}_k + \lambda_k q_k = \tilde{u}_k \tag{4}$$

with initial conditions

$$q_k(0) = q_k^0 = \int_\Omega w_0(x)\varphi_k(x)dx. \tag{5}$$

The constraint (3) in this case transforms into equivalent, condition

$$\sup_{x\in\Omega} \left|\sum \varphi_k(x)\tilde{u}_k(t)\right| \le v^0.$$

In this condition is replaced in by the stronger constraint

$$\sum_k \Phi_k |\tilde{u}_k(t)| \le v^0, \tag{6}$$

where $\Phi_k = \sup_{x\in\Omega}|\varphi_k(x)|$. But in order to reduce the problem into one dimensional systems, it was further replaced by a "Hilbert brick" $|\tilde{u}_k| \le U_k$, where the linear dimensions of U_k were chosen so that the inequality $\sum_k \Phi_k U_k \le v^0$ holds.

We call (4)-(6) the problem ω-problem. First of all, by linear change of coordinates we convert system (4) and constraint (6) into the form

$$\dot{y}_k + \lambda_k y_k = u_k, \sum_k |u_k| \le 1. \tag{7}$$

Then, we fix in time an integer positive number n and we investigate the following finite-dimensional problem:

$$\dot{y}_j + \lambda_j y_j = u_j, U = \left\{u \in R^n \middle| |u_1| + |u_2| + ... + |u_n| \le 1\right\}. \tag{8}$$

Unless otherwise stated, the subscript j will everywhere take the values $1,2,...,n.$

Note that problem (7)-(8) corresponds to a linear controlled system with a diagonal matrix, while the control region U is an n-dimensional octahedron with vertices $e_j^\sigma = \sigma(\delta_{j1},...,\delta_{j2},...,\delta_{jn}), \quad \sigma \in \{-1,+1\}$ where δ_{ji} is the Kronecker delta.

We apply Pontryagin’s maximum principle to time-optimal problem (7)-(8). Since $\lambda_k \ne \lambda_l$ when $k \ne l,$ the generecity condition is satisfied, so that the optimal control is unique and also piecewise-constant. Moreover, all the eigenvalues λ_j are real and non-negative, so that the optimal control exists for any initial point. Based on these properties, we will proceed to the construction of feedback control: in each interval of constancy, the optimal trajectory is determined from a system of the form

$$\dot{y}_j = -\lambda_j y_j \pm 1,\ \dot{y}_s = -\lambda_s y_s \ \ for\ s \ne j.$$

It is easily computed that the optimal trajectory $y(t)$, started from y^0, has switchings at the following instant of times: for $j = 1,2,...,n-1$

$$T_j = \frac{1}{\lambda_j}\ln\left(\lambda_j\left|y_j^0\right| + e^{\lambda_j T_{j-1}}\right) = \frac{1}{\lambda_j}\ln\left(\lambda_j\left|y_n^0\right|e^{-\lambda_n T_{j-1}} + 1\right) + T_{j-1}; \quad T_0 = 0, \quad (9).$$

Hence we have the property $y_j(T_j) = 0$ for the optimal trajectory.

Optimal control themselves also can be expressed simply in terms of T_j:

$$u_j(t) = \begin{cases} -sgn y_j^0 \ when \ \left[T_{j-1}, T_j\right), \\ 0 \ \ when \ t \in \left[0, T_j\right] \setminus \left[T_{j-1}, T_j\right). \end{cases}$$

In the next section it will be needed to extend $u(t)$ along the semiaxis $[0,+\infty)$, defined as $u(t) = 0$ when $t > T_{n-1}$.

Theorem 1. *If the limit $T_*(y^0)$ of recurrent equation (9) as $n \to \infty$ exists, then it is the time optimal transition time in the ω-time optimal problem. Morevoer, letting p be the least number such that $y_p^0 \ne 0$ for any initial point y^0 we have*

$$T_* \le \left|y_p^0\right| + \sum_{k=p+1}^{\infty} \left|y_k^0\right| e^{-\lambda_k \left|y_p^0\right|}.$$

Let $\ell^2 = \{= (y_1, y_2, \ldots) \mid y_n \in, \sum_{n\ge1} y_n^2 < \infty\}$. We consider ℓ^2 with it's natural norm: $\|y\|_2^2 = \sum_{n\ge1} y_n^2$, which turns it into a Hilbert space.

Given an infinite system of ODEs:

$$\dot{y}_n = \lambda y_n + y_{n+1},\ y_n(0) = y_{n,0},$$

where $\lambda \in \mathbb{R}$ is a fixed number and $y_0 = \{y_{n,0}\}_{n\in} \in \ell^2$. We can rewrite the system in an operator form $\dot{y} = A,\ \ y(0) = y_0,$ where $y_0 = \{y_{n,0}\}_n$ and $A : \ell^2 \to \ell^2$ is a linear operator defined by $A = \{\lambda y_n + y_{n+1}\}_{n\in\mathbb{N}}$.

This is an example of an ODE in a Banach space, which is a well studied topic, here we study the stability and control problems. In particular, we construct controls function explicitly. Observe that A is a bounded linear operator.

We also consider the Cauchy problem for non-homogeneous equation

$$\dot{y} = Ay + f,\ \ y(0) = y_0, \tag{10}$$

for $f :\to \ell^2$, $f \in L^2([0,T], \ell^2)$, i.e. $\| f\|_{L^2}^2 = \int_0^T \| f(t)\|_2^2\, dt < +\infty$[1].

A function $y:[0,T] \to \ell^2$ defined as

$$y(t) = e^{tA} y_0 + e^{tA} \int_0^t e^{-sA} f(s) ds$$

is called a mild solution of (10) if $y \in C([0,T], \ell^2)$. Here the integration is understood componentwise. For completeness we start with the following.

Proposition 1. *For every* $f \in L^2([0,T], \ell^2)$ *and* $y_0 \in \ell^2$ *we have* $y \in C([0,T], \ell^2)$.

The next result is about stability. In this simple setting we can characterize the system completely. We have the following.

Proposition 2. *Let* $y(t)$ *be the solution of* (10) *with an initial condition* $y_0 \in \ell^2$. *System* (10) is *asymptotically stable if and only if* $\lambda \le -1$. *Moreover for every* $y_0 \in \ell^2$ *and for every* $t \in \mathbb{R}$ *holds* $\| e^{tA} y_0 \|_2 \le e^{(1+\lambda)t} \|y_0\|_2$.

Let $\rho > 0$ be fixed. A control function $f : \mathbb{R} \to \ell^2$ is called admissible if

$$\| f\|_{L^2}^2 = \int_0^T \| f(t)\|_2^2\, dt \le \rho^2.$$

We say that the system (10) is *null-controllable* from $y_0 \in \ell^2$ an admissible control $f : \mathbb{R} \to \ell^2$ and $T = T(f) \in \mathbb{R}$ such that the solution of (10) satisfies $y(T) = 0$.

[1] We note that this norm coincides with $\| f\|_{L^2}^2 = \sum_{n\ge1} \int_0^T | f_n(t) |^2\, dt$ thanks to Beppo-Levi's theorem.

We say that the system (10) *is locally null-controllable* if there exists $\delta = \delta(\rho) > 0$ such that (10) is null-controllable from any $y_0 \in \ell^2$ with $\|y_0\| \le \delta$.

We say that the system (10) is *globally null-controllable* if it is null-controllable from any $y_0 \in \ell^2$.

The main result of this section is the following

Theorem 2. (i) *The system* (10) *is locally null-controllable for every* $\lambda \in \mathbb{R}$. (ii) *If* $\lambda \le -1$, *then system* (10) *is globally null-controllable.* (iii) *If* $\lambda < -1$ *the systems can be transferred from an initial point* $y_0 \in \ell^2$ *into the origin for time* $\tau \ge \|y_0\|_2^4 / \kappa\rho^4$, *where* κ *is a constant independent of* y_0.

Notice that for $-1 < \lambda < 0$ we construct solutions going to ∞ as $t \to +\infty$, i.e. 0 is not Lyapunov stable. Thus jump when passing from $\lambda = -1$ somewhat unusual. But apparently it is due to the structure of ℓ^2 and very special structure of $A = \lambda I + E$, i.e. the shift operator $E : \ell^2 \to \ell^2$ is weakly contracting ($E^n y \to 0$ as $n \to \infty$ for all $y \in \ell^2$) in this case. The proofs show that analogues results are true for all ℓ^p spaces with $1 \le p < +\infty$. However, in ℓ^∞ the trivial solution 0 is Lyapunov stable, but it is not asymptotically stable when $\lambda = -1$.

Remark 1. Consider the system $\dot{y} = Ay$, $y(0) = y_0 \in \ell^\infty$, where

$$\ell^\infty = \{y = (y_1, y_2, \dots) \mid \sup_{n\in\mathbb{N}} |y_n| < \infty\}.$$

Then $e = (1,1,1\dots) \in \ell^\infty$ is an eigenvector of e^{tE} corresponding to the eigenvalue e^t. Thus 0 is Lyapunov stable but it is not asymptotically stable.

To prove controllability, we use Gramian operators and prove an observability inequality. For $\tau \in \mathbb{R}$ define

$$W(\tau) = \int_0^\tau e^{-sA} \cdot e^{-sA^*} ds$$

We need to show that W is a self-adjoint operator and estimate the norm of its inversive, which will be deduced from the following standard

Lemma 1. *Let* $L : \mathcal{H} \to \mathcal{H}$ *be a self adjoint operator defined on a Hilbert space* $(\mathcal{H}, \|\cdot\|)$. *Assume that there exists* $\kappa > 0$ *such that* $\|Lx\| \ge \kappa \|x\|$ *for all* $x \in L$. *Then* L *is invertible and* $\|L^{-1}\| \le \kappa^{-1}$.

Let for every $i \in \mathbb{N}$, $d_i \times d_i$ matrices A_i be given with $2 \le d_i \le d$, where $d \ge 2$ is a fixed integer. We consider a differential game described by the following countable system of differential equations

$$\dot{x}_i = A_i x_i + u_i - v_i,\ x_i(0) = x_{i0} \in \mathbb{R}^{d_i},\ i = 1,2,..., \qquad (11)$$

where u_i is a control parameter of the pursuer, v_i is a control parameter of the evader, both assumed to be locally integrable functions with values in $\mathbb{R}^{d_i}$ for all $i \in \mathbb{N}$ satisfying certain constraints. For convenience we form column vector

$x=(x_1^*,x_2^*,\ldots)^*$, where x_i^* is the transpose of $x_i\in\mathbb{R}^{d_i}$, for $i=1,2,\ldots$[1]. Assume that $x_0=(x_{10},x_{20},\ldots,)\in\ell^2$, i.e. $\|x\|^2=\sum_{i=0}^{\infty}\|x_{i0}\|^2<+\infty$, where $\|x_{i0}\|$ is the Euclidean norm of $x_{i0}\in\mathbb{R}^{d_i}$. Let $A=\mathrm{diag}(A_1,A_2,...)$ be an operator whose action is defined as $Ax=\mathrm{diag}([A_1x_1]^*,[A_2x_2]^*,\ldots)$. We would like to define e^{tA} for suitable classes of matrices A_i. The problem here is that A is not necessarily defined on ℓ^2 or even on ℓ^∞ (i.e. Ax is not necessarily in ℓ^∞ for $x\in\ell^\infty$ since $|A_ix_i|$ may go to infinity as $n\to\infty$). Therefore, we have to justify the existence of solutions to the above Cauchy problem for initial points in ℓ^2.

We consider pursuit evasion differential game which consists of two separate problems as usual. The pursuit game can be completed in time $T>0$ provided there exists a control function of the pursuer $u:\mathbb{R}\to\ell^2$ such that for any control of $v:\mathbb{R}\to\ell^2$ the solution of $x:\mathbb{R}\to\ell^2$ of (11) for any x_0 satisfies $x(T)=0$. In this case T is called the guaranteed pursuit time. Below we state precise conditions that are imposed on u, v.

Along with (11) we also consider inhomogeniuous equation, where we have to justify the existence of solutions of the Cauchy problem

$$\dot{x}_i=A_ix_i+w_i,\ x_i(0)=x_{i0}\in\mathbb{R}^{d_i},\ i=1,2,..., \tag{12}$$

with $w_i:\mathbb{R}\to\mathbb{R}^{d_i}$ locally integrable. We look for solutions of (12) from the space of continuous functions $C([0,T];\ell^2)$ for some $T>0$, such that the coordinates $x_i(\cdot)$ of $x:[0,T]\to\ell^2$ are almost everywhere differentiable.

Definition 1. We say that a family of matrices $\{A_i\}_{i\in\mathbb{N}}$ is uniformly normalizable if there exists a family $\{P_i\}_{i\in\mathbb{N}}$ of non-singular matrices and a constant $C\ge 1$ such that $\|P_i\|\cdot\|P_i^{-1}\|\le C$ and $P_iA_iP_i^{-1}$ is a matrix in the Jordan normal form for all $i\in\mathbb{N}$.

Notice that there exist uniformly normalizable families of matrices, i.e., if all elements are already in Jordan normal form, then we may take $P_i=\mathrm{Id}$ for all $i\in\mathbb{N}$. On the other hand one can construct families, which aren't uniformly normalizable. In the simplest case, if we fix $\lambda_1,\lambda_2\in\mathbb{R}$ and set $A_i=\begin{pmatrix}\lambda_1 i & 0\\ 0 & \lambda_2/i\end{pmatrix}$ p-0-$=\|P\|^{-1}\cdot\|P\|=i^2$. This shows that the condition in Definition 1 is a restriction on the spectrum of A_i and for 2×2 matrices it implies that the ratio of eigenvalues of A_i remain bounded for all $i\ge 1$.

We assume that the family of matrices in (11) and (12) are uniformly normalizable and control parameters of the players satisfy the following constraint.

[1] Actually, we are just concatenating vectors x_1, x_2, $\ldots$ one below another to obtain an infinite vector. Below we adopt this point of view, which simplifies the notation considerably.

Definition 2. Fix $\theta > 0$ and let $B(\theta)$ be the set of all functions $w(\cdot) = \left(w_1(\cdot), w_2(\cdot), ...\right)$, $w:[0,T] \to \ell^2$, with measurable coordinates $w_i(\cdot) \in \mathbb{R}^{d_i}$, $0 \le t \le T$, $i = 1, 2, ...$, that satisfy the constraint $\sum_{i=1}^{\infty} \int_0^T \| w_i(s) \|^2 \, ds \le \theta^2$.

$B(\theta)$ is called the set of admissible control functions.

We have the following

Theorem 3. *Let* $\{A_i\}$ *be a family of uniformly normalizable matrices. If the real parts of eigenvalues of matrices* $A_1, A_2, ..$ *are negative, then for any* $w \in B(\theta)$, $\theta > 0$ *system* (10) *has a unique solution for any* $z_0 \in \ell^2$. *Moreover, the corresponding components of the solution* $x(t) = (x_1(t), x_2(t), \ldots)$ *are given by*

$$x_i(t) = e^{tA_i} x_{i0} + \int_0^t e^{(t-s)A_i} w_i(s) ds, \quad i \in \mathbb{N}.$$

Definition 3. System (12) is called globally asymptotically stable if $\lim_{t \to +\infty} x(t) = 0$ for a solution $x(t)$ of (12) with any initial condition $x_0 \in \ell^2$ and $w_i \equiv 0$ for all $i \in \mathbb{N}$. Further, system (12) is **null-controllable** from $x_0 \in \ell^2$ if there exists an admissible control $u \in B(\theta)$ and $T = T(u) \in \mathbb{R}_+$ such that the solution of (12) starting from x_0 satisfies $x(T) = 0$. We say that system (12) is *null-controllable in large* if it is null-controllable from any $x_0 \in \ell^2$. Also, $\inf_{u \in B(\theta)} T(u)$ is called **optimal time of translation** and $u \in B(\theta)$ realizing the minimum is called **time optimal control**.

Theorem 4. *Under the assumptions of Theorem 3 system* (12) *is globally asymptotically stable and null controllable in large. Time optimal control exists and can be constructed explicitly.*

Further, we consider a pursuit-evasion differential game (11). Fix $\rho, \sigma > 0$. A function $u(\cdot) \in B(\rho)$ ($v(\cdot) \in B(\rho)$) is called an admissible control of the pursuer (evader).

Definition 4 A function $u:[0,T] \times \ell^2 \to \ell^2$ with coordinates $u_k(t) = v_k(t) + \omega_k(t)$, $\omega \in B(\rho - \sigma)$, which is an admissible control of the pursuer for every $v \in \ell^2$ is called a strategy of the pursuer.

Theorem 5. *Suppose that* $\rho > \sigma$ *and the assumptions of Theorem 3 are satisfied. Then, for any admissible control of the evader* v *there exists a strategy of the pursuer* u *and* $\vartheta_1 > 0$ *such that the solution of* (11) *satisfies* $z(\tau) = 0$ *for some* $0 \le \tau \le \vartheta_1$, *i.e., the game* (11) *can be completed within time* ϑ_1.

Let $x_1, \ldots, x_m$, $m \ge 1$, be the points moving in $\mathbb{R}^n$ whose dynamics are described by the equations

$$\dot{x}_i = -\lambda_i x_i + v - u_i, \quad x_i(0) = x_i^0, \quad i = 1, 2, \ldots, m, \tag{13}$$

where $u_1, \ldots, u_m$ are the control parameters of pursuers and v is that of evader, $\lambda_i > 0$, $x_i, x_i^0, u_i, v \in \mathbb{R}^n$, $n \ge 2$, $x_i^0 \ne 0$, $i = 1, ..., m$.

Definition 5. *Measurable functions* $u_i(t)$ *and* $v(t)$, $t \ge 0$, *that satisfy the following integral constraints*

$$\int_0^\infty |u_i(t)|^2\, dt \le \rho_i^2, i = 1,\ldots m; \int_0^\infty |v(t)|^2\, dt \le \sigma^2. \qquad (14)$$

are called controls of the i th pursuer and evader, respectively.

Theorem 6 *If* $\rho_1^2 + \ldots + \rho_m^2 \le \sigma^2$, *then evasion is possible in game* (13) – (14), *i.e. there exists a strategy* V *which is explicitly constructed of evader such that for any controls of pursuers* $x_i(t) \ne 0$ *for some* $i = 1,\ldots,m$, *and all* $t \ge 0$.

The second chapter of the dissertation **"Stability of the statistical properties for the Lorenz system"** and is devoted to the study the statistical stability and continuity of the vairance in the central limit theorem for the Lorenz system. In 1963 Lorenz introduced the following system of equations

$$\begin{cases} \dot{x} = -10x + 10y \\ \dot{y} = 28x - y - xz \\ \dot{z} = -8/3\, z + xy \end{cases} \qquad (15)$$

as a simplified model for atmospheric convection. Numerical analysis performed by Lorenz showed that the above system exhibits sensitive dependence on initial conditions and has a non-periodic "strange" attractor. A rigorous mathematical framework of similar flows was initiated with the introduction of the so called geometric Lorenz flow. Nowadays it is well known that the geometric Lorenz attractor, whose vector field will be denoted by X_0, is robust in the C^1 topology. This means that vector fields X_ε that are sufficiently close in the C^1 topology to X_0 admit invariant contracting foliations $\mathcal{F}_\varepsilon$ on the Poincaré section Σ and they admit strange attractors. More precisely, there exists an open neighbourhood U in $\mathbb{R}^3$ containing Λ, the attractor of X_0, and an open neighbourhood $\mathcal{U}$ of X_0 in the C^1 topology such that for all vector fields $X_\varepsilon \in \mathcal{U}$, the maximal invariant set $\Lambda_{X_\varepsilon} = \cap_{t\ge 0} X_\varepsilon{}^t(U)$ is a transitive set which is invariant under the flow of X_ε .

We consider perturbations of X_0 which are consistent with the results of. From now on we are going to set $p := 1/\alpha$. Let $\mathcal{X}$ be the family of C^1 perturbations of X_0; i.e., there exists an open neighbourhood U in $\mathbb{R}^3$ containing Λ, the attractor of X_0, and an open neighbourhood $\mathcal{U}$ of X_0 containing $\mathcal{X}$ such that

(1) for each $X_\varepsilon \in \mathcal{X}$, the maximal forward invariant set Λ_{X_ε} is contained in U and is an attractor containing a hyperbolic singularity;

(2) for each $X_\varepsilon \in \mathcal{X}$, Σ is a cross-section for the flow with a return time τ_ε and a Poincaré map F_ε;

(3) for each $X_\varepsilon \in \mathcal{X}$, the map F_ε admits a $C^{1+\alpha}$ uniformly contracting invariant foliation $\mathcal{F}_\varepsilon$ on Σ;

(4) $F_\varepsilon : \Sigma \to \Sigma$ is given by[1]

$$F_\varepsilon(x,y) = (T_\varepsilon(x), g_\varepsilon(x,y));$$

(5) the map $T_\varepsilon : I \to I$ is transitive piecewise C^1 expanding with two branches and a discontinuity point O_ε such that $T(O_\varepsilon^{+}) = -1/2$, $T(O_\varepsilon^{-}) = 1/2$ and $\lim_{x \to O_\varepsilon^{\pm}} T'_\varepsilon(x) = +\infty$.

Let V be a neighbourhood of 0. Let $(X_\varepsilon)_{\varepsilon \in V}$ be a family of flows which is endowed with some topology $\mathfrak{T}$. Assume that every X_ε admits a unique SRB measure μ_ε. The family $(X_\varepsilon)_{\varepsilon \in V}$ is called statistically stable if $\varepsilon \mapsto X_\varepsilon$ is continuous at $\varepsilon = 0$ in the weak $*$-topology, i.e.

$$\lim_{\varepsilon \to 0} \int f d\mu_\varepsilon = \int f d\mu_0,$$

for any continuous function $f : \mathbb{R}^3 \to \mathbb{R}$. Statistical stability is defined analogously for discrete-time dynamical systems. We refer the reader to the articles for more information.

Theorem 7. *Let $X_\varepsilon \in \mathcal{X}$. Then*

1) *X_ε admits a unique invariant probability SRB measure μ_ε.*

2) *For any continuous $\varphi : \mathbb{R}^3 \to \mathbb{R}$ we have*

$$\lim_{\varepsilon \to 0} \int \varphi d\mu_\varepsilon = \int \varphi d\mu;$$

where μ is the SRB measure associated with X_0; i.e. the family $\mathcal{X}$ is statistically stable.

Remark 2. 1) in Theorem 7. is known. See for instance. We prove 2). Let $\psi : \mathbb{R}^3 \to \mathbb{R}$. For $\varepsilon \geq 0$, define

$$\Psi_\varepsilon(\xi) := \int_0^{\tau_\varepsilon(\xi)} \psi(X_\varepsilon(t))dt,$$

where $\xi := (x, y, 1) \in \Sigma \setminus \Gamma_\varepsilon$. The next main result of this section is the following

Theorem 8. *Let $\psi : \mathbb{R}^3 \to \mathbb{R}$ be Hölder.*

1) Then

$$\frac{1}{\sqrt{t}} \left(\int_0^t \psi \circ X_\varepsilon(s) ds - t \int \psi d\mu_\varepsilon \right) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2_{X_\varepsilon})$$

with

$$\sigma^2_{X_\varepsilon}(\psi_\varepsilon) = \sigma^2_{F_\varepsilon}(\Psi_\varepsilon) \Big/ \int \tau_\varepsilon d\mu_{F_\varepsilon}.$$

2) *Moreover, for $\psi \in C^1$ we have* $\lim_{\varepsilon \to 0} |\sigma^2_{X_\varepsilon}(\psi) - \sigma^2_X(\psi)| = 0.$

To prove 2) of Theorem 2.1.2 we use the its one dimensional version, which requires introduction of some Banach spaces.

We say $f : I \to \mathbb{R}$ is a function of universally bounded p-variation if

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[1] By c) g_ε is $C^{1+\alpha}$. Moreover, it is a uniform contraction on stable leaves.

$$V_p(f) := \sup_{-1/2 \le x_0 < \ldots < x_n \le 1/2} \left(\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p \right)^{1/p} < +\infty.$$

The space of universally bounded p-variation functions is denoted by $UBV_p(I)$ and it will play a key role in studying perturbations of X_0.

Let $S_\rho(x) := \{y \in I : |x - y| < \rho\}$ and $f : I \to \mathbb{R}$ be any function defined on I. Let

$$osc(f, \rho, x) := ess\sup\{|f(y_1) - f(y_2)| : y_1, y_2 \in S_\rho(x)\},$$

and $osc_1(f, \rho) = \| osc(f, \rho, x) \|_1$, where the essential supremum is taken with respect to the two dimensional Lebesgue measure on $I \times I$ and $\| \cdot \|_1$ is the L^1- norm with respect to Lebesgue measure on I. Fix $\rho_0 > 0$ and let $BV_{1,1/p} \subset L^1$ be the Banach space equipped with the norm

$$\| f \|_{1,1/p} = V_{1,1/p}(f) + \| f \|_1, \quad V_{1,1/p}(f) = \sup_{0 < \rho \le \rho_0} \frac{osc_1(f, \rho)}{\rho^{1/p}}.$$

Notice that $V_{1,1/p}(\cdot)$ depends on ρ_0. It is known that that $BV_{1,1/p}$ is a Banach space. Moreover, the unit ball of $BV_{1,1/p}$ is compact in L^1 and for any $p \ge 1$ and $f \in UBV_p(I)$ we have $V_{1,1/p}(f) \le 2^{1/p} V_p(f)$.

Theorem 9. *Let T_ε be the family of the 1-d maps corresponding to the family $\mathcal{X}$. Then*

1) T_ε satisfies a CLT with variance $\sigma^2_{T_\varepsilon}$; i.e., for $\overline{\psi}_\varepsilon : I \to \mathbb{R}$, $\overline{\psi}_\varepsilon \in BV_{1,1/p}$ with $\int \overline{\psi}_\varepsilon d\overline{\mu}_\varepsilon = 0$, where $\overline{\mu}_\varepsilon$ is the T_ε $(\varepsilon \ge 0)$ absolutely continuous invariant measure

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \overline{\psi}_\varepsilon (T_\varepsilon^i x) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2_{T_\varepsilon})$$

and $\sigma^2_{T_\varepsilon} > 0$ if and only if $\overline{\psi}_\varepsilon \ne c + \nu \circ T - \nu$, $\nu \in BV_{1,1/p}$, $c \in \mathbb{R}$;

2) Let $\overline{\psi}_\varepsilon, \overline{\psi} \in BV_{1,1/p}$ such that $\| \overline{\psi}_\varepsilon \|_\infty$ is uniformly bounded in ε and $\int | \overline{\psi}_\varepsilon - \overline{\psi} | dx \to 0$ as $\varepsilon \to 0$ [1]. Then

$$\lim_{\varepsilon \to 0} | \sigma^2_{T_\varepsilon}(\overline{\psi}_\varepsilon) - \sigma^2_T(\overline{\psi}) | = 0.$$

The third chapter of the dissertation **"Linear response for non-uniformly expanding random maps"** is devoted to the smoothness of the stationary measure with respect to external parameter.

A random dynamical system on a measurable space $(X, \mathcal{B})$ is defined by a skew product map $S : \Omega \times X \to \Omega \times X, S(\omega, x) = (\sigma\omega, T_{\sigma(\omega)}(x))$, where $\sigma : \Omega \to \Omega$

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[1] For instance this assumption is automatically satisfied when $\overline{\psi}_\varepsilon := \hat{\psi} - \int \hat{\psi} d\overline{\mu}_\varepsilon$ and $\overline{\psi} := \hat{\psi} - \int \hat{\psi} d\overline{\mu}$ for some $\hat{\psi} \in BV_{1,1/p}$.

is a map preserving a probability measure $\mathbb{P}$, T_ω is called a fibre map. There are two approaches to study random dynamical systems: annealed and quenched. In the annealed case we are interested in studying statistical properties of a stationary measure, which is a measure on the phase space X. In this case we study averaged dynamical properties over ω.

Let $T_\omega : X \to X$, $\omega \in \Omega$ be a family of maps such that for each $\omega \in \Omega$, there exists a finite or countable set Z_ω and a partition (mod 0) of X into open intervals $X_{z,\omega}$, $z \in Z_\omega$ such that the restriction of T_ω to $X_{z,\omega}$ is C^3 and onto. We denote by $g_{z,\omega}$ the inverse branches of T_ω on $X_{z,\omega}$.

Below, we look at the random dynamical system as a Markov process with transition kernel

$$p(x, A) = \int_\Omega 1_A(T_\omega(x)) d\mathbb{P}(\omega).$$

We say that a measure μ on X is stationary if for any measurable $A \subset X$

$$\int_X p(x, A) d\mu(x) = \mu(A),$$

We set $L\Phi := \int_\Omega L_{T_\omega} \Phi d\mathbb{P}(\omega)$. where L_{T_ω} is the transfer operator associated with the map T_ω, defined by

$$L_{T_\omega}\Phi = \sum_{z\in Z} \Phi \circ g_{z,\omega} \cdot | g_{z,\omega}'|.$$

In particular, any stationary measure μ absolutely continuous with respect to m, with density h, satisfies $L_{\mathbb{P}} h = h$. $L_{\mathbb{P}}$ is called the transfer operator of this random dynamical system.

Let $\mathbb{P}_\epsilon$ be a family of probability measures on Ω. We are interested in studying the change in the statistical behaviour of the random system[1] $(\Omega, \{T_\omega\}, \mathbb{P}_\epsilon)$ as ε changes in a neighborhood V of 0. The transfer operator of the perturbed system is denoted by $L_{\mathbb{P}_\epsilon}$. We assume:

Assumption A. (A1) there exists $\tilde{D} > 0$ such that

$$\left| \frac{g_{z,\omega}'(x)}{g_{z,\omega}'(y)} - 1 \right| \le \tilde{D} | x - y | \tag{16}$$

for any $x, y \in X$, $z \in Z$ and $\omega \in \Omega$. Moreover, there exists $M > 0$ independent of ε such that for $i = 2,3$ we have

$$\sup_{\varepsilon\in V} \sum_{z\in Z} \sup_{x\in X} \int_\Omega | g_{z,\omega}^{(i)} | d\mathbb{P}_\varepsilon(\omega) \le M. \tag{17}$$

(A2) There exists $\beta \in (0,1)$ such that $\sup_{\omega\in\Omega} \sup_{z\in Z} \sup_{x\in X} | g_{z,\omega}'(x) | \le \beta$.

[1] A common way to have a parameter dependent random system is also when the system consists of a fixed probability space $(\Omega, \mathbb{P})$ and a parametrized family of maps $T_{\omega,\varepsilon}$, $\omega \in \Omega$, $\varepsilon \in V$. This situation can be represented in our framework, with the new probability space $\Omega \times V$ and the probability measure $\mathbb{P}_\varepsilon = \mathbb{P} \otimes \delta_\varepsilon$.

Proposition 3. *Under assumptions (A1)-(A2), for each* $\varepsilon \in V$ *the operators* $L_{\mathbb{P}_\varepsilon}$ *has a spectral gap on* C^1 *and* C^2. *In particular, the random dynamical system* $(\Omega, \{T_\omega\}, \mathbb{P}_\varepsilon)$ *admits a unique stationary density* $h_\varepsilon \in C^2$.

For each $z \in Z$, and $\Phi \in L^1(X)$, let $\psi_z(,x) = \int_\Omega [\Phi \circ g_{z,\omega} | g'_{z,\omega}|](x) d\ (\omega)$.

Assumption B.

• Assume that $L_{\mathbb{P}_\varepsilon}$ admits a uniform spectral gap on C^1. Moreover, assume that the random dynamical system $(\Omega, \{T_\omega\}, \mathbb{P}_\varepsilon)$ admits a unique stationary density $h_\varepsilon \in C^2$.

• For $\Phi = h_0 \in C^2$ assume that the partial derivatives $\partial_\varepsilon \psi_z(,x)$, $\partial_x \psi_z(\varepsilon, x)$, $\partial_x \partial_\varepsilon \psi_z(,x)$, $\partial_\varepsilon \partial_x \psi_z(,x)$ exist and are jointly continuous in (ε, x) on $X \times V$. Hence, $\partial_\varepsilon \partial_x \psi_z(,x) = \partial_x \partial_\varepsilon \psi_z(,x)$. Moreover, assume that for $i = 0,1$ we have

$$\sum_{z \in Z} \sup_{\in V} \sup_{x \in X} |\partial_\varepsilon \psi_z^{(i)}(\varepsilon, x)| < \infty,$$

where $\psi_z^{(0)} = \psi_z$ and $\psi_z^{(1)} = \partial_x \psi_z$.

• In addition, assume that for any $\Phi \in C^1$, $\psi_z(,x)$ and $\partial_x \psi_z(,x)$ exists and are jointly continuous. Moreover, for $i = 0,1$, assume that $\sum_{z \in Z} \sup_{\varepsilon \in V} \sup_{x \in X} |\psi_z^{(i)}(\varepsilon, x)| < \infty$.

Theorem 10. *Let* $(\Omega, \{T_\omega\}, \mathbb{P}_\varepsilon)$ *be a family of random dynamical systems as described above. Under assumption B, the density* h *of the stationary measure is differentiable as a* C^1 *element at* $\varepsilon = 0$, *that is there exists* $h^* \in C^1$ *such that*

$$\left\| \frac{h_\varepsilon - h_0}{\varepsilon} - h^* \right\|_{C^1} \to 0. \tag{18}$$

In addition, the following linear response formula holds:

$$h^* := (I - L_{P_0})^{-1} \partial_\varepsilon L_{P_\varepsilon} h_0 |_{\varepsilon=0}, \tag{19}$$

where $\partial_\varepsilon L_{P_\varepsilon} h_0 |_{\varepsilon=0} = \partial_\varepsilon \sum_{z \in Z} \int_\Omega [h_0 \circ g_{z,\omega} | g_{z,\omega}' |] dP_\varepsilon(\omega)$.

Below we develop a general framework to study linear response for random compositions of maps where inducing is required

Given $\omega = \Omega^N$, we write $T^n_{\hat{\omega}} := T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} \circ T_{\omega_0} : X \to X$. For $\omega \in \Omega^N$ we define $\hat{T}_{\hat{\omega}}$ as the first return map under the orbit of $T^n_{\hat{\omega}}$ to Δ; i.e., for $x \in \Delta$

$$\hat{T}_{\hat{\omega}}(x) = T^{R_{\hat{\omega}}(x)}_{\hat{\omega}}(x), \ R_{\hat{\omega}}(x) = \inf\{n \geq 1 : T^n_{\hat{\omega}}(x) \in \Delta\}.$$

Let Z be the set of finite sequences of the form $z = z_0 z_1 \dots z_n$, where $z_0 \in S^{in}$ and $z_i \in S^{out}$ for $i = 1, \dots, n$ and $n \in \mathbb{N}$. We denote by $|z| = n+1$ the length of the word $z \in Z$. We set $g_{z,\hat{\omega}} = g_{z_0,\omega_0} \circ g_{z_1,\omega_1} \circ \cdots \circ g_{z_n,\omega_n}$. Then for $x \in X$ we have $T^{n+1}_{\hat{\omega}} \circ g_{z,\hat{\omega}}(x) = x$. For each $\hat{\omega}$, the cylinder sets $g_{z,\hat{\omega}}(\Delta)$, form a partition of Δ

(mod 0). Note that on $g_{z,\hat{\omega}}(\Delta)$ we have $R_{\hat{\omega}}(\cdot)=n+1$. We assume that the maps $(\Delta,\{\hat{T}_{\hat{\omega}}\}_{\hat{\omega}\in\Omega})$ satisfy the assumptions A and B, piecewise C^3 and piecewise onto.

An induced random dynamical system. Let $\hat{\Omega}=\Omega^N$ and $\hat{\mathbb{P}}=\mathbb{P}^N$. We study the random dynamical system defined by the i.i.d. composition of maps $\hat{T}_{\hat{\omega}}$, with $\hat{\omega}$ distributed according to $\hat{\mathbb{P}}$. Following the framework of theroem 10, for $\Phi\in L^1(\Delta)$, the transfer operator of this induced random system is given by

$$\hat{L}_{\hat{\mathbb{P}}}\Phi=\int_{\hat{\Omega}}\hat{L}_{\hat{\omega}}\Phi d\hat{\mathbb{P}}(\hat{\omega})=\sum_{z\in Z}\int_{\hat{\Omega}}\Phi\circ g_{z,\hat{\omega}}\,|\,g'_{z,\hat{\omega}}\,|\,d\hat{\mathbb{P}}(\hat{\omega}),$$

where $\hat{L}_{T_{\hat{\omega}}}$ is the transfer operator associated with $\hat{T}_{\hat{\omega}}$. Notice that $\hat{L}_{\hat{\mathbb{P}}}$ reduces to

$$\hat{L}_{\hat{\mathbb{P}}}\Phi=\sum_{z\in Z}\int_{\Omega}\Phi\circ g_{z,\omega}\,|\,g'_{z,\omega}\,|\,d\hat{\mathbb{P}}(\omega).$$

The density $\hat{h}$ of any absolutely continuous stationary measure $\hat{\mu}$ of the induced random system satisfies $\hat{L}_{\hat{\mathbb{P}}}\hat{h}=\hat{h}.$

Unfolding the density of the induced random dynamical system: For $\Phi\in L^1(\Delta)$, let $F_{\mathbb{P}}(\Phi):X\to\mathbb{R}$ be defined by

$$F_{\mathbb{P}}(\Phi):=1_\Delta\Phi+(1-1_\Delta)\sum_{z\in Z}\int_{\hat{\Omega}}\Phi\circ g_{z,\hat{\omega}}\,|\,g_{z,\hat{\omega}}'\,|\,d\hat{\mathbb{P}}(\hat{\omega}).$$

Note that $F_{\mathbb{P}}$ is a linear operator. In the next lemma we show that if $\hat{h}$ is a stationary density for the induced random system then $F_{\mathbb{P}}\hat{h}$ is a stationary density for the original system.

Propostion 4. *Let* $\hat{h}\in L^1(\Delta)$ *be such that* $\hat{L}_{\hat{\mathbb{P}}}\hat{h}=\hat{h}$. *Then* $L_{\mathbb{P}}(F_{\mathbb{P}}\hat{h})=F_{\mathbb{P}}\hat{h}$.

The perturbed random system: Let $\mathbb{P}_\varepsilon$ be a family of probability measures on Ω, supported on $\bigcup_{k=1}^{\ell}\{k\}\times I_k$. Let V be a neighbourhood of 0. Let π_ε be the marginal measure of $\mathbb{P}_\varepsilon$ on $\{1,\dots,\ell\}$ and $\eta_{k,\varepsilon}$ be the conditional measure of $\mathbb{P}_\varepsilon$ on $I_k\times\{k\}$. As above, we are interested in studying the change in the statistical behaviour of the random system $(\Omega,\{T_\omega\},\mathbb{P}_\varepsilon)$ as ε changes. We assume that $\hat{L}_{\hat{\mathbb{P}}_\varepsilon}$ admits a uniform spectral gap on C^1. Moreover, assume that the random dynamical system $(\hat{\Omega},\{\hat{T}_{\hat{\omega}}\},\hat{\mathbb{P}}_\varepsilon)$ admits a unique stationary density $\hat{h}_\varepsilon\in C^2$. By Proposition 4, the stationary densities of the original random system $(\Omega,\{T_{\hat{\omega}}\},\mathbb{P}_\varepsilon)$ and the induced one $(\hat{\Omega},\{\hat{T}_{\hat{\omega}}\},\hat{\mathbb{P}}_\varepsilon)$ are related by[1]

$$h_\varepsilon=F_{\mathbb{P}_\varepsilon}(\hat{h}_\varepsilon).$$

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[1] Note that h_ε is un-normalized. When h_ε is integrable; i.e., when the random system preserves a probabilistic acsm, once the derivative of h_ε is obtained, the derivative of the normalized density can be easily computed. Indeed, if $h_\varepsilon=h+\varepsilon h^*+o(\varepsilon)$, then $\int h_\varepsilon=\int h+\int\varepsilon h^*+o(\varepsilon)$. Thus, $\partial_\varepsilon(h_\varepsilon\big/\int h_\varepsilon)|_{\varepsilon=0}=h^*-h\int h^*$.

Now we need to introduce a Banach space that contains the stationary density of the original random dynamical system. In general, one should tailor the Banach space norm according to information on the regularity of the stationary density of the original random dynamical system[1].

Let $\mathcal{H}$ denote the set of continuous functions on $(0,1]$ with the norm

$$\| f \|_{\mathcal{H}} = \sup_{x\in(0,1]} | x^{\gamma} f(x) |,$$

for a fixed $\gamma \ge 0$. When equipped with the norm $\|\cdot\|_{\mathcal{H}}$, $\mathcal{H}$ is a Banach space. For each $z \in Z$, and $\Phi \in L^1(X)$, let

$$\hat{\psi}_z(,x) = \int_{\hat{\Omega}} [\Phi \circ g_{z,\hat{\omega}} \,|\, g_{z,\hat{\omega}}'|](x) d\hat{\mathbb{P}}_{\varepsilon}(\hat{\omega}).$$

In Theorem 11 below we prove the differentiability, at $\varepsilon = 0$, of $\varepsilon \mapsto h_{\varepsilon}$ as an element of $\mathcal{H}$ under the following set of conditions:

• For $\Phi = \hat{h}_0 \in C^2$ we assume that the partial derivatives $\partial_{\varepsilon}\hat{\psi}_z(\varepsilon,x)$, $\partial_x\hat{\psi}_z(\varepsilon,x)$, $\partial_x\partial_{\varepsilon}\hat{\psi}_z(\varepsilon,x)$, $\partial_{\varepsilon}\partial_x\hat{\psi}_z(\varepsilon,x)$ exist and are jointly continuous on $\Delta \times V$, whence, satisfy the commutation relation $\partial_{\varepsilon}\partial_x\hat{\psi}_z(\varepsilon,x) = \partial_x\partial_{\varepsilon}\hat{\psi}_z(\varepsilon,x)$. For any $\Phi \in C^1$, $\hat{\psi}_z(\varepsilon,x)$ and $\partial_{\varepsilon}\hat{\psi}_z(\varepsilon,x)$ exist and are jointly continuous on $(0,1]\times V$. Moreover we assume that for $i = 0,1$ we have

$$\sum_{z\in Z} \sup_{\varepsilon\in V} \sup_{x\in\Delta} | \partial_{\varepsilon}\hat{\psi}_z^{(i)}(\varepsilon,x) | < \infty,$$

where $\hat{\psi}_z^{(0)} = \hat{\psi}_z$ and $\hat{\psi}_z^{(1)} = \partial_x\hat{\psi}_z$.

• For $\Phi \in C^1$, $\psi_z(,x)$ and $\partial_x\psi_z(,x)$ exists and are jointly continuous. Moreover, for $i = 0,1$ we assume that

$$\sum_{z\in Z} \sup_{\varepsilon\in V} \sup_{x\in\Delta} | \hat{\psi}_z^{(i)}(\varepsilon,x) | < \infty.$$

• For any $\Phi \in C^0$, we assume that

$$\sum_{z\in Z} \sup_{\varepsilon\in V} \| \hat{\psi}_z(\varepsilon,\cdot) \|_{\mathcal{H}} < \infty.$$

• For $\Phi = \hat{h} \in C^2$ we assume that

$$\sum_{z\in Z} \sup_{\varepsilon\in V} \| \partial_{\varepsilon}\hat{\psi}_z(\varepsilon,\cdot) \|_{\mathcal{H}} < \infty.$$

Theorem 11. *Let $(\Omega, \{T_{\omega}\}, \mathbb{P}_{\varepsilon})$ be a family of random dynamical systems defined as above Then*

(1) *There exists $h^* \in \mathcal{H}$ such that*

$$\lim_{\varepsilon\to 0} \| \frac{h_{\varepsilon} - h_0}{\varepsilon} - h^* \|_{\mathcal{H}} = 0;$$

i.e., h_{ε} is differentiable as an element of $\mathcal{H}$ with respect to ε at $\varepsilon = 0$.

(2*) In particular,under the above condition if $\gamma < 1$ in definition of $\mathcal{H}$, then*

[1] For instance, if the stationary density has a singularity at $x = 1$, one can modify the dentition of $\mathcal{H}$ below accordingly. Or, if the stationary density is known to be C^0, then one can simply work with C^0.

$$\lim_{\varepsilon\to 0}\| \frac{h_\varepsilon - h_0}{\varepsilon} - h^* \|_1 = 0.$$

The fourth chapter is devoted to quenched decay of correlations and titled **"Non-uniformly hyperbolic random dynamical systems"**, where we study random dynamical systems for almost every realisation and obtain upper estimates for the rates of decay of correlations.

Definition 6. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces on $\Omega\times X$ and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\rho_n = 0$. We say that f_ω admits *quenched decay of correlations* at rate ρ_n if for P-almost all ω and for any $\varphi\in\mathcal{B}_1$ and $\psi\in\mathcal{B}_2$ there are constants C_ω and $C_{\varphi,\psi}$ such that

$$|Cor_{n,\omega}^{(f)}(\varphi,\psi)| \le C_\omega C_{\varphi,\psi}\rho_n, \quad |Cor_{n,\omega}^{(p)}(\varphi,\psi)| \le C_\omega C_{\varphi,\psi}\rho_n.$$

Remark 3. Note that if C_ω is P-integrable, then this implies the same rate for the integrated correlations; i.e., $\int_\Omega Cor_{n,\omega}^{(f)}(\varphi,\psi)dP \le \hat{C}_{\varphi,\psi}\rho_n$. The importance of knowing the rate of the integrated correlations is due to its relation to the annealed correlations of the skew product. Indeed, setting $\bar{\varphi}_\omega := \int_X \varphi d\mu_\omega$ and $\bar{\psi}_\omega := \int_X \psi d\mu_\omega$ we have $Cor_n(T,\varphi,\psi) = \int_\Omega Cor_{n,\omega}^{(f)}(\varphi,\psi)dP + Cor_n(\sigma,\bar{\varphi},\bar{\psi})$.

Below we develop a framework for obtaining quneched decay ofcorrelartionfor weakly hyperbolic system. The abstract scheme is called random Young towers.

Let $\Lambda\subset X$ be a measurable set with $m(\Lambda)=1$. Consider a family of maps $f_\omega : X\to X$, where f_ω depends only on the zeroth coordinate of ω. We say that f_ω admits a random tower on $\Lambda\subset X$ if for almost every $\omega\in\Omega$ there exists a countable partition $\{\Lambda_j(\omega)\}_j$ of Λ and a measurable return time function $R_\omega : \Lambda\to\mathbb{N}$ that is constant on each $\Lambda_j(\omega)$ such that $f_\omega^{R_\omega}(x) = f_{\sigma^{R_\omega(x)-1}\omega}\circ\cdots\circ f_{\sigma\omega}\circ f_\omega(x)\in\Lambda$ for P-almost every $\omega\in\Omega$ and m-almost every $x\in\Lambda$. Given the above information we define a random tower for almost every ω as

$$\Delta_\omega = \left\{(x,\ell)\in\Lambda\times\mathbb{Z}_+ \mid x\in\cup_j\Lambda_j(\sigma^{-\ell}\omega), j,\ell\in\mathbb{Z}_+, 0\le\ell\le R_{\sigma^{-\ell}\omega}(x)-1\right\}$$

and random tower map $F_\omega : \Delta_\omega\to\Delta_{\sigma\omega}$ by

$$F_\omega(x,\ell) = \begin{cases} (x,\ell+1), & \text{if} \quad \ell+1 < R_{\sigma^{-\ell}\omega}(x) \\ (f_{\sigma^{-\ell}\omega}^{\ell+1}x, 0), & \text{if} \quad \ell+1 = R_{\sigma^{-\ell}\omega}(x). \end{cases}$$

We assume that the random tower satisfies the following properties.

(P1) **Markov**: for each $\Lambda_j(\omega)$ the map $F_\omega^{R_\omega}|\Lambda_j(\omega):\Lambda_j(\omega)\to\Lambda$ is a bijection;

(P2) **Bounded distortion**: There are constants $D>0$ and $0<\gamma<1$ such that for all ω and each $\Lambda_j(\omega)$ the map $F_\omega^{R_\omega}|\Lambda_j(\omega)$ and its inverse are non-singular with respect to m with corresponding Jacobian $JF_{\omega}^{R_\omega}|\Lambda_j(\omega)$ which is positive and for each $x,y\in\Lambda_j(\omega)$ satisfies the following

$$\left|\frac{JF_{\omega}^{R_\omega}(x)}{JF_{\omega}^{R_\omega}(y)}-1\right|\le D\gamma^{s(F_{\omega}^{R_\omega}(x,0),F_{\omega}^{R_\omega}(x,0))};$$

(P3) **Weak expansion**: $\mathcal{P}_\omega$ is a generating partition for F_ω i.e. diameters of the partitions $\vee_{j=0}^{n}F_{\omega}^{-j}\mathcal{P}_{\sigma^j\omega}$ converge to zero as n tends to infinity;

(P4) **Return time asymptotics**: There are constants $C>0$, $a>1$, $b\ge 0$, $u>0$, $v>0$, a full measure subset $\Omega_1\subset\Omega$ and a random variable $n_1:\Omega_1\to\mathbb{N}$ such that

$$\begin{cases} m\{x\in\Lambda\,|\,R_\omega(x)>n\}\le C(\log n)^b/n^a, \text{ when ever } n\ge n_1(\omega), \\ P\{n_1(\omega)>n\}\le Ce^{-un^v}; \end{cases}$$

(P5) **Aperiodicity**: There are $N\in\mathbb{N}$ and $\{t_i\in\mathbb{Z}_+\,|\,i=1,2,...,N\}$ such that g.c.d.$\{t_i\}=1$ and $\epsilon_i>0$ so that for almost every $\omega\in\Omega$ and $i=1,2,\ldots N$ we have $m\{x\in\Lambda\,|\,R_\omega(x)=t_i\}>\epsilon_i$.

(P6) **Finiteness**: There exists an $M>0$ such that $E(m(R>n))\le M$ for all $\omega\in\Omega$

(P7) **Annealed return time asymptotics**: There are constants $C>0$, $\hat{b}\ge 0$ and $a>1$ such that $\int_\Omega m\{x\in\Lambda\,|R_\omega=n\}dP\le C\dfrac{(\log n)^{\hat{b}}}{n^{a+1}}$.

Definition 7.

(i) The fibered system $(F,\nu)=(F_\omega,\nu_\omega)_{\omega\in\Omega}$ is exact iff $\vee_{n=0}^{\infty}F^{-n}\mathcal{B}$ is trivial; i.e., for any $B\in\vee_{n=0}^{\infty}F^{-n}\mathcal{B}$, either for almost all ω, $\nu_\omega(B)=0$ or for almost all ω, $\nu_\omega(B)=1$.

(ii) The random skew product (F,ν) is mixing iff for all $\varphi,\psi\in L^2(\nu)$,

$$\lim_{n\to\infty}\left|\int_\Omega\int\varphi_{\sigma^n\omega}\circ F_\omega^n\cdot\psi_\omega d\nu_\omega dP-\int_\Omega\int\varphi_\omega d\nu_\omega dP\int_\Omega\int\psi_\omega d\nu_\omega dP\right|=0.$$

To state the results we need to introduce some function spaces on Δ, which are necessary to state the theorems. Below we let constants $u>0$, $v>0$, $a>1$, $b\ge 0$, $0<\gamma<1$ be as in (P2) and (P4) above and set

$$F_\gamma^+=\{\varphi_\omega:\Delta_\omega\to\mathbb{R}\,|\,\exists C_\varphi>0,\ \forall I_\omega\in_\omega P_\omega,\ \text{either } \varphi_\omega|I_\omega\equiv 0$$
$$\text{or } \varphi_\omega|I_\omega>0 \text{ and } \left|\log\frac{\varphi_\omega(x)}{\varphi_\omega(y)}\right|\le C_\varphi\gamma^{s(x,y)},\ \forall x,y\in I_\omega\}.$$

Let $K_\omega:\Omega\to\mathbb{R}_+$ be a random variable with $\inf_\Omega K_\omega>0$ and

$$P\{\omega\,|\,K_\omega>n\}\le e^{-un^v}.$$

Define the space of random bounded functions as

$$L_{\infty}^{K_\omega} = \{\varphi_\omega : \Delta_\omega \to \mathbb{R} \mid \exists C'_\varphi > 0,\ \sup_{x\in\Delta_\omega} |\varphi_\omega(x)| \le C'_\varphi K_\omega\}$$

and a space of random Lipschitz functions

$$F_\gamma^{K_\omega} = \{\varphi_\omega \in L_\infty^{K_\omega} \mid \exists C_\varphi > 0, |\varphi_\omega(x) - \varphi_\omega(y)| \le C_\varphi K_\omega \gamma^{s(x,y)},\ \forall x, y \in \Delta_\omega\}.$$

The first result is the existence of absolutely continuous sample measures.

Theorem 12. *For almost every* $\omega \in \Omega$ *there is a family* $\nu_{\sigma^k\omega}$ *which is equivariant and absolutely continuous with respect to* m*:* $(F_{\sigma^k\omega})_* \nu_{\sigma^k\omega} = \nu_{\sigma^{k+1}\omega}$*, with* $\nu_\omega = h_\omega m$*. Moreover, there exists tempered function* K_ω *such that* $h_\omega \in F_\gamma^{K_\omega} \cap F_\gamma^+$ *for almost every* $\omega \in \Omega$*.*

The next main result is about the decay of future and past quenched correlations. Let $\delta > 0$.

Theorem 13. *Let* $\delta > 0$*.* K_ω *be the function given in Theorem 12. There exits a full measure set* $\Omega_0 \subset \Omega$ *and a random variable* C_ω *on* Ω_0 *such that for every* $\varphi \in L_\infty^{K_\omega}$*,* $\psi \in F_\gamma^{K_\omega}$ *there exits a constant* $C_{\varphi,\psi}$ *such that for every* $\omega \in \Omega_0$

(i) *“Future" operational correlations :*

$$\left| \int (\varphi_{\sigma^n\omega} \circ F_\omega^n) \psi_\omega dm - \int \varphi_{\sigma^n\omega} d\nu_{\sigma^n\omega} \int \psi_\omega dm \right| \le C_\omega C_{\varphi,\psi} n^{1+\delta-a}.$$

(ii) *“Past" operational correlations :*

$$\left| \int (\varphi_\omega \circ F_{\sigma^{-n}\omega}^n) \psi_{\sigma^{-n}\omega} dm - \int \varphi_\omega d\nu_\omega \int \psi_{\sigma^{-n}\omega} dm \right| \le C_\omega C_{\varphi,\psi} n^{1+\delta-a}.$$

Moreover, there exist constants $C > 0$*,* $u' > 0$*,* $v' \in (0,1)$ *such that*

$$P\{C_\omega > n\} \le C e^{-u' n^{v'}}.$$

CONCLUSION

Current dissertation is devoted to the study of various notions of stability for dynamical systems: asymptotic stability, statistic stability and linear response and decay of correlations, i.e. asymptotic stability of the measure describing the system. Briefly the main results of the current dissertation are the following

We have shown that stability in Banach spaces depend on the norm. In a simple example with large spectrum we have shown that in infinite dimensional spaces new phenomena can occur. But if the system is "well" approximated with operators, whose spectrum is discrete then the situation with stability and controllability resembles the finite dimensional case.

In the example of Lorenz flow we show that if the system and its perturbations preserve some hyperbolicity then it is possible to prove the statistical stability. Further, if the degree of hyperbolicity can be quantified, e.g. with derivates with respect to the parameter, then linear response holds.

Further we developed an abstract machinery to obtain decay of corrrelations for random compositions of maps from some weakly hyperbolic family maps. These methods are aseffective as its deterministic analogies and applicable to prototypical exmaples of non-uniformly hyperbolic random dynamical system. In case the noise in the system is i.i.d., the most chaotic system in the family dominate the statistic properties.

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ИНСТИТУТЕ МАТЕМАТИКИ ИМЕНИ В.И.РОМАНОВСКОГО**

ИНСТИТУТ МАТЕМАТИКИ

РУЗИБАЕВ МАРКС БОТИРБОЕВИЧ

УСТОЙЧИВОСТЬ, ЭРГОДИЧНОСТЬ И УПВРАВЛЯЕМОСТЬ ДИНАМИЧЕСКИХ СИСТЕМ

01.01.02 – Дифференциальные уравнения и математическая физика

**АВТОРЕФЕРАТ
ДОКТОРСКОЙ (DSc) ДИССЕРТАЦИИ ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ**

ТАШКЕНТ–2024

Тема докторской (DSc) диссертации зарегистрирована в Высшей аттестационной комиссии при Министерстве Высшего образования, Науки и Инноваций Республики Узбекистан за № B2024.3DSc/FM274.

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Научный консультант:	**Азамов Абдулла Азамович** доктор физико-математических наук, академик
Официальные оппоненты:	**Джалилов Ахтам Абдурахманович** доктор физико-математических наук, профессор
	Дурдиев Дурдимурад Каландарович доктор физико-математических наук, профессор
	Сачков Юрий Леонидович доктор физико-математических наук, профессор
Ведущая организация:	**Национальный университет Узбекистана**

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С диссертацией можно ознакомиться в Информационно-ресурсном центре Института Математики имени В.И.Романовского (регистрационный номер № 192). (Адрес: 100174, г.Ташкент, Алмазарский район, ул. Университетская. Тел.: (+99871)-207-91-40).

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O'ZBEKISTON RESPUBLIKASI FANLAR AKADEMIYASI V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI

У.А.Розиков
Председатель Научного совета по присуждению ученых степеней, д.ф.-м.н., академик.

Ж.К. Адашев
Ученый секретарь Научного совета по присуждению ученых степеней, д.ф.-м.н., старший научный сотрудник.

Р.Р.Ашуров
Заместитель председателя Научного семинара при Научном совете по присуждению ученых степеней, д.ф.-м.н., профессор.

ВВЕДЕНИЕ (аннотация докторской диссертации)

Цель исследований заключается в решении задачи управляемости, доказательстве статистической устойчивости, непрерывности, гладкости и асимптотической устойчивости инвариантных мер динамических систем при возмущении.

Объект исследования: Линейные, управляемые и слабо гиперболические хаотические динамические системы.

Научная новизна исследования:

доказана асимптотическая устойчивость и ноль–управляемость нового класса управляемых систем, определенных в бесконечномерных пространствах;

решена задача преследования-убегания в динамических играх с интегральными ограничениями, определенных в конечномерных пространствах;

доказана статистическая устойчивость, а также устойчивость дисперсии геометрического аттрактора Лоренца;

разработан абстрактный метод доказательства гладкости стационарных мер случайных динамических систем. С помощью этого метода доказана гладкость стационарной меры отображении Гаусса-Реньи и Манневиля-Помеа;

разработан аналог абстрактной башни Янга для оценки скорости убывание корреляционной функции в почти всех реализациях случайных динамических систем и получены оценки скорости убывания корелляционной функция для возмущения слабо гиперболических случайных систем.

Доказано существование критической прерывистости для системы, полученной случайными композициями отображений из логистической семьи.

Внедрение результатов исследования.

Разработанный абстрактный метод для доказательства гладкости стационарных мер случайных динамических систем применена в международном проекте IP-2019-04-1239 на тему "Трансферные операторы, предельные законы и бесконечномерные динамические системы" для решения задачи линейного отклика в случайных динамических системах (справка Университета Риеки, Хорватия, 21 ноября 2023 года). Применение научных результатов позволила решить задачу линейного отклика в случайных слабо-гиперболических системах.

Результаты по статистической устойчивости аттрактора Лоренца и результаты по управляемым системам, определенных в бесконечномерных пространствах, были использованы в проекте OT-F4-(36+32) на тему "Разработка новых методов решения задач математической физики и оптимального управления. Неклассические начальные и краевые задачи для уравнений с дискретными производными и их приложения" (справка Национального университета Узбекистана, №04/11-9455 от 28 декабря 2023 года). Применение результатов позволила построить синтез оптимального управления процессов, описывающих бесконечную систему дифференциальных уравнений.

Абстрактный метод, разработанный для оценки скорости убывания корреляции в слабо гиперболических случайных системах, использован в научных статьях, опубликованных в международных научных журналах, а именно при оценке скорости убывания корреляций (Advances in Math, 2023, том 426; Ann. Probab. 2022, 50 (1) 241–303; Commun. Math. Phys. 2021, 385, 905–935). Применение научных результатов позволило получить оценки скорости убывания корреляций в системах со слабой гиперболичностью.

Апробация результатов исследования: Результаты исследования были обсуждены на восьми международных научно-практических конференциях, проведенных в ведущих научных центрах мира.

Структура и объём диссертации: Диссертация состоит из введения, четырёх глав, заключения и библиографии. Объём работы−197 страниц.

E’LON QILINGAN ISHLAR RO‘YHATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (part 1, часть 1)

1. J.F. Alves, W. Bahsoun, M. Ruziboev, P. Varandas: Quenched decay of correlations for nonuniformly hyperbolic random maps with an ergodic driving system // *Nonlinearity*, 36 3294-3318, (2023) (3. Scopus. IF=1,36).

2. J.F. Alves, W. Bahsoun, M. Ruziboev: Almost sure rates of mixing for partially hyperbolic attractors. // *Journal of Differential Equations*, 311, 98-157, (2022) (3. Scopus. IF=2,05).

3. A. Azamov, G. Ibragimov, K. Mamayusupov, M. Ruziboev: On the Stability and Null-Controllability of an Infinite System of Linear Differential Equations. // *Journal of Dynamical and Control Systems*, 29, 595-605 (2023) (3. Scopus. IF=0,6).

4. A.A. Azamov, M.B. Ruziboev: The time-optimal problem for evolutionary partial differential equations. // *Journal of Applied Mathematics and Mechanics*. 77, 220-224, (2013).

5. W. Bahsoun, I. Melbourne, M. Ruziboev: Variance continuity for Lorenz flows. // Annales Henri Poincare, 21(6), 1873-1892, (2020) (3. Scopus. IF=1,01).

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9. A.J. Homburg, C. Kalle, M. Ruziboev, E. Verbitskiy, B. Zeegers: Critical intermittency in random interval maps, // *Communications in Mathematical Physics*, 394, 1–37, (2022). (3. Scopus. IF=1,61).

10. G. Ibragimov, M. Ferrara, M. Ruziboev, A.B. Pansera: Linear evasion differential game of one evader and several pursuers with integral constraints. // *International Journal of Game Theory*, 50(3), 729-750, (2021), (3. Scopus. IF=0,6).

11. M. Ruziboev: Almost Sure Rates of Mixing for Random Intermittent Maps. // In Differential Equations and Dynamical Systems. USUZCAMP 2017. *Springer Proceedings in Mathematics and Statistics*, V. 268. Springer, Cham (2018). (3. Scopus IF 0.17).

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13. S. Luzzatto, M. Ruziboev: Statistical properties of direct product systems. // *Discrete and Continuous Dynamical Systems*, 36(3), 1465-1491, (2016). (3. Scopus IF 1,1).

II bo'lim (part 2, часть 2)

14. M. Muhammad, M. Ruziboev: Quenched Mixing Rates for Doubly Intermittent Maps // *arXiv:2404.09751*.

15. A. Larkin, M. Ruziboev: Quenched decay of correlations for random contracting Lorenz maps // *arXiv:2308.04351*.

16. M. I. Akhmedov, D. Babajanov, M. Ruziboev: Linearized Korteweg - De Vries equation on a tree with unbounded root and edges. // *arxiv.org/abs/2106.11939.*

17. M. Ruziboev: Quenched decay of correlations for random maps with an er godic driving system. // *BudWiSer: Budapest-Vienna seminar on Ergodic Theory* September 23, Hungary (2022), math.bme.hu/~dinamika/Ruziboev_szept23.txt.

18. M. Ruziboev: Quenched decay of correlations for random maps with an er godic driving system. // *Invited speaker, Limit Theorems for Slowly Mixing Systems*, ICMS workshop, 06-10 June, Edinburgh, UK, (2022), Limit Theorems for Slowly Mixing Systems | ICMS - International Centre for Mathematical Sciences.

19. M. Ruziboev: Critical intermittency in random interval maps. // *Invited speaker, Dynamics Days Europe 2021*, 23-27 August, Nice, France (2021), Random dynamical systems: statistical aspects.

20. M. Ruziboev: Almost sure rates of mixing for random partially hyperbolic maps. // *SIAM Conference on Applications of Dynamical Systems*, 23-27 May, (2021), Conference Program Online

21. M. Bekimov, M. Ruziboev, I. Zaynabiddinov: Pursuit evasion differential games in l^p on a finite time interval, Invited speaker. // *Dynamic Systems: Stability, Control, Differential Games* 9-13 September, Ekaterinburg, RF, (2024), pp. 433-436.

22. M.Ruziboev: Linear response for random dynamical systems. // *Invited speaker of Dynamical systems and beyond*, 25-27 March, Pisa, Italy (2019), AbstractTitoli.pdf, p.7.

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25. M. Ruziboev, On the time optimal problem for heat equation. // *Control and Optimization of Dynamical System*, September 27 - October 1, Tashkent, Uzbekistan (2009), pp. 138-139.

Avtoreferat “O‘zbekiston matematika jurnali” tahririyatida 2024 yil 11-noyabrda tahrirdan o‘tkazilib, o‘zbek, rus va ingliz tillaridagi matnlar o‘zaro muvofiqlashtirildi.

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