V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.02/30.12.2019.FM.86.01 RAQAMLI ILMIY KENGASH

CHIRCHIQ DAVLAT PEDAGOGIKA UNIVERSITETI

QARSHIBOYEV OBID SHERQUL O‘G‘LI

PANJARADA ANIQLANGAN BA’ZI MODELLARNING GIBBS O‘LCHOVLARI UCHUN CHEGARAVIY KONFIGURATSIYALAR

01.01.01 –Matematik analiz

FIZIKA-MATEMATIKA FANLARI BO‘YICHA

FALSAFA DOKTORI (PhD) DISSERTATSIYASI

AVTOREFERATI

Toshkent – 2025

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Fizika-matematika fanlari bo‘yicha falsafa doktori (PhD) dissertatsiyasi avtoreferati mundarijasi

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Fizika-matematika fanlari bo'yicha falsafa doktori (PhD) dissertatsiyasi mavzusi O'zbekiston Respublikasi Oliy ta'lim, fan va innovatsiyalar vazirligi huzuridagi Oliy attestatsiya komissiyasida B2021.3.PhD/FM622 raqam bilan ro'yxatga olingan.

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KIRISH (falsafa doktori (PhD) dissertatsiyasi annotatsiyasi)

Dissertatsiya mavzusining dolzarbligi va zarurati. Jahon miqyosida olib borilayotgan fizik, biologik sistemalarning termodinamik xossalarini oʻrganishga bagʻishlangan bir qancha ilmiy-amaliy masalalar koʻpincha statistik mexanika modellarini tadqiq qilishga keltiriladi. Gibbs oʻlchovlari statistik mexanika modellarining faza almashishlari nazariyasini tadqiq qilish uchun asosiy vosita hisoblanadi. Shuningdek, kimyo, xizmat koʻrsatish nazariyasi, materialshunoslik kabi fan va texnikaning turli sohalaridagi muammolarni hal qilishda Gibbs oʻlchovlarini qoʻllash muhim ahamiyat kasb etmoqda. Statistik mexanikaning klassik modellari uchun barcha Gibbs oʻlchovlari toʻplamini toʻliq tavsiflash masalasi hech bir model uchun bugungi kungacha toʻla hal qilinmagan. Shu bois, bunday modellar uchun Gibbs oʻlchovlari toʻplamini boyitish, jumladan, panjaralarda aniqlangan ba’zi modellarning Gibbs oʻlchovlari uchun chegaraviy konfiguratsiyalar toʻplamini topish masalasi dolzarb masalalardan hisoblanadi.

Hozirgi kunda panjarali sistemalar uchun tashqi maydonli Hamiltonianlarga mos translyatsion-invariant hamda davriy Gibbs oʻlchovlarini mavjudligini koʻrsatish va bunday Gibbs oʻlchovlari toʻplamini toʻla tavsiflash dolzarb masalalardan biridir. Berilgan fizik model uchun Gibbs oʻlchovlari toʻplamida translyatsion-invariant va davriy oʻlchovlarning yagona emasligini aniqlash statistik fizikaning faza almashishlari nazariyasida muhim ahamiyat kasb etmoqda. Bu borada berilgan Hamiltonian uchun barcha translyatsion-invariant va davriy Gibbs taqsimotlarini topish, limit Gibbs oʻlchovlari translyatsion-invariant splitting Gibbs oʻlchovlari bilan mos keladigan chegaraviy konfiguratsiyalar toʻplamini tasniflash maqsadli ilmiy tadqiqotlardan hisoblanadi.

Mamlakatimizda taraqqiyotning asosiy tayanchlaridan biri boʻlgan fundamental fanlarning ilmiy va amaliy tatbiqiga ega boʻlgan sohalarga e’tibor kuchaytirilmoqda. Jumladan, oxirgi yillarda Keli daraxtida aniqlangan statistik mexanikaning klassik modellari uchun translyatsion-invariant, davriy va davriy boʻlmagan Gibbs oʻlchovlari mavjudligini aniqlash hamda bunday oʻlchovlar toʻplamini boyitish borasida salmoqli natijalarga erishildi. “Algebra va uning tatbiqlari, differensial tenglamalar va uning tatbiqlari, chiziqsiz sistemalar, dinamik sistemalar, matematik modellashtirish, stoxastik tahlil, funksional analiz, tibbiy-biologik informatika, hisoblash matematikasi” fanlarining ustuvor yoʻnalishlari boʻyicha xalqaro standartlar darajasida ilmiy tadqiqotlar olib borish matematika fanining asosiy vazifalari va faoliyat yoʻnalishlari etib belgilandi[1]. Qaror ijrosini ta’minlash uchun ilmiy natijalardan ilm-fanning turdosh sohalarida foydalanish maqsadida statistik mexanikaning klassik modellari uchun faza almashishlar nazariyasini rivojlantirish muhim ahamiyatga ega.

Oʻzbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PQ-4947-son “Oʻzbekiston Respublikasini yanada rivojlantirish boʻyicha harakatlar strategiyasi

[1] Oʻzbekiston Respublikasi Prezidentining 2019-yil 9-iyuldagi №PQ-4387 “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qoʻllab-quvvatlash, shuningdek, Oʻzbekiston Respublikasi Fanlar akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari toʻgʻrisida”gi qarori.

to‘g‘risida”gi va 2022-yil 28-yanvardagi PF-60-son “2022-2026-yillarga mo‘ljallangan Yangi O‘zbekistonning Taraqqiyot strategiyasi to‘g‘risida”gi Farmonlari, 2017-yil 17-fevraldagi PQ-2789-son “Fanlar akademiyasi faoliyati, ilmiy-tadqiqot ishlarini tashkil etish, boshqarish va moliyalashtirishni yanada takomillashtirish chora-tadbirlari to‘g‘risida”gi, 2020-yil 7-maydagi PQ-4708-son “Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida”gi, 2022-yil 28-yanvardagi PF-60-son “2022-2026-yillarga mo‘ljallangan Yangi O‘zbekistonning taraqqiyot strategiyasi to‘g‘risida”gi qarorlari hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalarni rivojlantirishning ustivor yo‘nalishlariga bog‘liqligi. Mazkur tadqiqot respublika fan va texnologiyalar rivojlanishining IV. “Matematika, mexanika va informatika” ustuvor yo‘nalishi doirasida bajarilgan.

Muammoning o‘rganilganlik darajasi.

Statistik mexanikaning asosiy masalalaridan biri berilgan sistemaning termodinamik muvozanatini tavsiflovchi Gibbs o‘lchovlarini tadqiq qilishdan iborat. Gibbs o‘lchovi ehtimollik o‘lchovi bo‘lib, u atrof muhit bilan termodinamik muvozanatda bo‘lgan sistemalarning holatini aniqlash maqsadida J.Gibbs tomonidan kiritilgan. Keyinchalik soha rivojlanib, alohida Gibbs o‘lchovlari nazariyasi shakllandi. Uning rivojlanishida E. Izing, R. Dobrushin, O. Lenford, D. Ryuel, R. Minlos R. Bekster, H.O. Georgi, V.A. Malishev, R.A. Minlos, K. Preston, Ya.G. Sinay, G. Gallavotti, F. Bonettolarning beqiyos xizmatlari asos bo‘ldi.

Klassik statistik mexanika modellarida Gibbs o‘lchovlari E.I. Dinaburg, Yu.M. Suhov, A.E. Mazel, L. Kokvil, K. Kuelske, A. Le Ny, D. Gandolfo, N.N. G‘anixo‘jaev, O‘.A. Roziqov, F.M. Muhamedov, M.M. Raxmatullayev, R.M. Xakimov, G‘.I. Botirov, O.N. Hakimov, N.M. Xatamov, Sh.A. Shoyusupov, F.H. Haydarov, M.A. Rasulova, A.M. To‘xtabayev, M.T. Mahammadaliyev, B.U. Abrayev va boshqa ko‘plab tadqiqotchilar tomonidan keng o‘rganilgan.

A.E. Mazel va Yu.M. Suhovning ishida, $\mathbb{Z}^d$ panjaradagi SOS (solid-on-solid) modeli qaralgan. Bu ishda past haroratlarda termodinamik fazalar tuzilmasini SOS modelining ustuvor asosiy holatlari orqali aniqlaydigan haroratning kritik nuqtasi mavjudligi isbotlangan. Keyinchalik, E.I. Dinaburg va A.E. Mazel ushbu panjarada tashqi maydonli SOS modelini o‘rganib, faza almashishlari sodir bo‘ladigan shartlarni topdilar.

O‘.A. Roziqov va Yu.M. Suhovning ishida, Keli daraxtida SOS modeli qaralgan. Bu ishda uch holatli model uchun kritik haroratdan past haroratlarda kamida uchta translyatsion-invariant splitting Gibbs o‘lchovlari mavjudligi isbotlangan. Bundan tashqari, ushbu ishda davriy Gibbs o‘lchovlarining to‘liq tavsifi berilgan. O‘.A. Roziqov ikkinchi tartibli Keli daraxtida chekli holatli SOS modelini kontur argumenti (Pirogov-Sinay nazariyasi) yordamida o‘rgangan. Bu usul yordamida, yetarlicha past haroratlarda qaralayotgan model uchun Gibbs o‘lchovlarining mavjudligi va ularning soni kamida spin qiymatlar soniga teng bo‘lishi isbotlangan. Keyinchalik, G‘.I. Botirov va O‘.A. Roziqov bu natijalarni ixtiyoriy tartibli Keli daraxti uchun umumlashtirdilar.

K. Kuelske va O‘.A. Roziqov uch holatli SOS modelini ikkinchi tartibli Keli daraxtida ko‘rib chiqdilar. Ular qaralayotgan modelning barcha translyatsion-invariant splitting Gibbs o‘lchovlarini tasnifladilar. Bundan tashqari, bu o‘lchovlar mavjud bo‘lganda chekka yoki chekka bo‘lmaslik shartlarini tekshirdilar. M.M. Raxmatullayev va B.U. Abrayev ixtiyoriy tartibli Keli daraxtida uch holatli SOS modelini o‘rganishdi va ma’lum translyatsion-invariant Gibbs o‘lchovlaridan foydalangan holda, ba’zi translyatsion-invariant bo‘lmagan Gibbs o‘lchovlarini qurishdi.

Y. Higuchi chegaraviy shartlarni o‘zgartirish orqali Keli daraxtida Ising modeli uchun limit Gibbs o‘lchovlarini qurish usulini ishlab chiqdi. Ushbu yondashuv sistemaning turli xil muvozanat holatlariga mos keladigan turli Gibbs o‘lchovlarini qurishning aniq yo‘lini taqdim etdi. D. Gandolfo, M.M. Raxmatullayev·va O‘.A. Roziqov ikkinchi tartibli Keli daraxtida chekli spin qiymatli Potts modelining translyatsion-invariant splitting Gibbs o‘lchovlari uchun chegaraviy shartlar qurdilar.

Shuni alohida ta’kidlab o‘tish kerakki, ko‘plab ilmiy ishlarga qaramasdan, Keli daraxtida birorta model uchun barcha limit Gibbs o‘lchovlarining to‘liq tavsifi hali olinmagan.

Dissertatsiya mavzusining dissertatsiya bajarilgan oliy ta’lim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bog‘liqligi. Dissertatsiya tadqiqoti Chirchiq davlat pedagogika universitetining ilmiy-tadqiqot ishlari rejasining “Fundamental tadqiqotlar” tarmog‘i doirasida bajarilgan.

Tadqiqotning maqsadi Keli daraxtida tashqi maydonli uch holatli solid-on-solid (SOS) modeli uchun translyatsion-invariant, davriy Gibbs o‘lchovlarini o‘rganish, limit Gibbs o‘lchovlariga mos chegaraviy konfiguratsiyalarni hosil qilish va faza almashishlarini o‘rganishdan iborat.

Tadqiqotning vazifalari:

Ikkinchi tartibli Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant splitting Gibbs o‘lchovlari to‘plamini to‘la tavsiflash, ixtiyoriy tartibli Keli daraxtida bunday o‘lchovlarning yagona bo‘lmaslik shartlarini topish;

Ixtiyoriy tartibli Keli daraxtida translyatsion-invariant tashqi maydonli uch holatli SOS modeli uchun davriy Gibbs o‘lchovlarining to‘la tavsiflash va translyatsion-invariant bo‘lmagan, ikki davriy Gibbs o‘lchovini mavjudligini ko‘rsatish;

Ixtiyoriy tartibli Keli daraxtida ikki davriy tashqi maydonli uch holatli SOS modeli uchun faza almashish mavjudligini ko‘rsatish;

Limit Gibbs o‘lchovlari translyatsion-invariant splitting Gibbs o‘lchovlari bilan mos keladigan chegaraviy konfiguratsiyalar to‘plamini tasniflash.

Tadqiqotning obyekti: Keli daraxtida tashqi maydonli uch holatli SOS modeli.

Tadqiqot predmeti. Gruppalar va graflar nazariyasi, Gibbs o‘lchovlari nazariyasi, algebra va sonlar nazariyasi, nochiziqli diskret dinamik sistemalar nazariyasi, nochiziqli Markov jarayonlari.

Tadqiqotning usullari. Tadqiqot ishida sonlar nazariyasi, funksional analiz, kombinatorika, gruppalar nazariyasi, o‘lchovlar nazariyasi, chiziqli algebra va dinamik sistemalar nazariyasi usullaridan foydalanilgan.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

Ikkinchi tartibli Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant splitting Gibbs o‘lchovlari to‘plami to‘la tavsiflandi, ixtiyoriy tartibli Keli daraxtida bunday o‘lchovlarning yagona bo‘lmaslik shartlari topildi;

Ixtiyoriy tartibli Keli daraxtida translyatsion-invariant tashqi maydonli uch holatli SOS modeli uchun davriy Gibbs o‘lchovlari to‘la tavsiflandi hamda translyatsion-invariant bo‘lmagan, ikki davriy Gibbs o‘lchovining mavjudligi ko‘rsatildi;

Ixtiyoriy tartibli Keli daraxtida ikki davriy tashqi maydonli uch holatli SOS modeli uchun faza almashish mavjudligi ko‘rsatildi;

Limit Gibbs o‘lchovlari translyatsion-invariant splitting Gibbs o‘lchovlari bilan mos keladigan chegaraviy konfiguratsiyalar to‘plami tasniflandi.

Tadqiqotning amaliy natijasi quyidagilardan iborat:

Yuqori tartibli tenglamaning barcha musbat ildizlari to‘plamining to‘liq tavsifi statistik mexanikaning tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant splitting Gibbs o‘lchovlari sonini aniqlash uchun qo‘llanildi. Tashqi maydonli uch holatli SOS modeli uchun davriy Gibbs o‘lchovlari mavjudligini isbotlash uchun funksiya qo‘zg‘almas nuqtalarining xarakterini aniqlaydigan usullar ishlatilgan.

Tadqiqot natijalarining ishonchliligi.

Natijalar matematik analiz, funksional analiz, tasodifiy jarayonlar uchun Markov nazariyasi, ehtimollar nazariyasi, diskret dinamik sistemalar va o‘lchovlar nazariyasi usullaridan foydalanib olingan. Olingan natijalar qat’iy matematik mulohazalarga asoslanib isbotlangan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot ishi natijalarining ilmiy ahamiyati fan va texnika sohalarini rivojlantirishda statistik mexanika masalalarini tadqiq etishda Gibbs o‘lchovlari nazariyasi asosiy vosita ekanligi bilan izohlanadi.

Tadqiqot natijalarining amaliy ahamiyati fizik sistemalar holatining o‘zgarishi tadqiq qilinganligi hamda aloqa tarmoqlari va neyron tarmoqlari nazariyasining ba’zi masalalarini yechish imkonini berganligi bilan izohlanadi.

Tadqiqot natijalarining joriy qilinishi. Dissertatsiya tadqiqoti davomida olingan ilmiy natijalar quyidagi ilmiy-tadqiqot loyihalariga tatbiq etilgan:

Keli daraxtida tashqi maydonga ega bo‘lgan uch holatli solid-on-solid (SOS) modeli uchun translyatsion-invariant, davriy Gibbs o‘lchovlari haqidagi natijalardan G0003247 raqamli “Panjarali modellarning renormalizatsiyalangan guruhlari bilan bog‘liq xaotik va aralash *p*-adik dinamik sistemalar” mavzusidagi xorijiy grant loyihasida daraxtlarda SOS va boshqa turdosh modellar uchun chegaraviy konfiguratsiyalarni tadqiq qilishda foydalanilgan (Birlashgan Arab Amirliklari universitetining 2024-yil 26-sentyabrdagi ma’lumotnomasi, BAA). Ushbu ilmiy natijaning qo‘llanilishi daraxtlarda SOS va boshqa turdosh modellar uchun chegaraviy konfiguratsiyalar to‘plamini tasniflash imkonini bergan.

Ixtiyoriy tartibli Keli daraxtida uch holatli translyatsion-invariant tashqi maydonli solid-on-solid (SOS) modeli uchun davriy Gibbs o‘lchovlari to‘plamini to‘liq tavsiflash metodologiyasidan, shuningdek, limit Gibbs o‘lchovlari translyatsion-invariant splitting Gibbs o‘lchovlari bilan mos keladigan chegaraviy konfiguratsiyalar to‘plami tasnifidan 374874-2022 fundamental loyihasida “Faza almashish muammolari va kritik hodisalar. Ular tenglamalarining matematik nuqtayi nazari, tez almashishlar va asimptotikalar” nomli fizik tizimlarda fazaviy o‘tish mavjudligini va ushbu tizimning dinamik xususiyatlarini o‘rganishda foydalanilgan (O‘sh Davlat universitetining 2024-yil 5-noyabrdagi 1429-son guvohnomasi, Qirg‘iziston). Ushbu ilmiy natijani qo‘llash diskret vaqtli dinamik tizimlarning asimptotik xususiyatlarini va davriy trayektoriyalarini o‘rganish imkonini berdi.

Tadqiqot natijalarining aprobatsiyasi. Dissertatsiyaning asosiy natijalari 7 ta ilmiy-amaliy anjumanlarda, jumladan 3 tasi xalqaro va 4 tasi respublika miqyosidagi anjumanlarda muhokamadan o‘tkazilgan.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha jami 12 ta ilmiy ish chop etilgan, shulardan, 5 tasi O‘zbekiston Respublikasi Oliy Attestatsiya Komissiyasining doktorlik dissertatsiyalari asosiy ilmiy natijalarini chop etish tavsiya etilgan ilmiy nashrlarda, jumladan 4 tasi xorijiy va 1 tasi respublika jurnallarida nashr etilgan.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish qismi, uchta bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan tashkil topgan. Dissertatsiyaning hajmi 98 betni tashkil etadi.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida tadqiqot mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan, dissertatsiya mavzusi bo‘yicha xorijiy ilmiy tadqiqotlarning tahlili berilgan, muammoning o‘rganilganlik darajasi yoritilgan, tadqiqotning maqsad va vazifalari, obyekti va predmeti ko‘rsatilgan, tadqiqot natijalarining ilmiy yangiligi ochib berilgan, olingan natijalarning nazariy va amaliy ahamiyati ko‘rsatilgan, tadqiqot natijalarining tatbiqi, shuningdek, nashr etilgan ilmiy ishlar va dissertatsiyaning tuzilishi haqida ma’lumotlar keltirilgan.

Dissertatsiyaning “**Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant splitting Gibbs o‘lchovlari**” deb nomlanuvchi birinchi bobida tartibi ikkidan katta Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant splitting Gibbs o‘lchovlarida olingan natijalar ifoda etilgan.

$\Gamma^k = (V, L)$ bu tartibi $k \geq 1$ va markazi x^0 bo‘lgan cheksiz Keli daraxti, ya’ni har bir uchidan $k+1$ ta qirra chiquvchi siklga ega bo‘lmagan, bog‘langan va yo‘naltirilmagan cheksiz graf. Bu yerda V grafning uchlari to‘plami va L uning qirralari to‘plamidir. Agar x va y uchlarni tutashtiruvchi $<x, y> \in L$ qirra mavjud bo‘lsa, bu uchlarga yaqin qo‘shnilar deyiladi. $<x, x_1>, \ldots <x_{d-1}, y>$ turli qirralar ketma-ketligiga esa x uchdan y uchga olib boruvchi yo‘l deyiladi. Yo‘l yordamida daraxtda metrika aniqlash mumkin, ya’ni x uchdan y uchga olib boruvchi yo‘ldagi qirralar soniga shu uchlar orasidagi masofa deyiladi va u $d(x, y)$ kabi belgilanadi.

Quyidagi $V_n = \{x \in V : d(x, x^0) \leq n\}$, $W_n = \{x \in V : d(x, x^0) = n\}$ to‘plamlarni qaraylik. $x \in W_n$ uchun $S(x) = \{y \in W_{n+1} : d(y, x) = 1\}$ to‘plamni aniqlaymiz. $S(x)$ to‘plamga x uchning to‘g‘ri avlodlari to‘plami deyiladi.

$A \subseteq V$ uchun σ_A konfiguratsiya $x \in A \to \sigma_A(x) \in \Phi = \{0, 1, ..., m\}$ kabi aniqlanadigan funksiyadir. Barcha konfiguratsiyalar to‘plami $\Omega_A = \Phi^A$ bilan ustma-ust tushadi. Biz soddalik uchun $\Omega = \Omega_V$ va $\sigma = \sigma_V$ belgilashlardan foydalanamiz.

Tashqi maydonli SOS modeli quyidagi formal Hamiltonian bilan aniqlanadi

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)| - \sum_{x \in V} \alpha_{\sigma(x), x}, \tag{1}$$

bunda $J \in \mathbb{R}$ ta’sirlashish o‘zgarmasi, $\alpha_x = (\alpha_{0,x}, \alpha_{1,x}, ..., \alpha_{m,x}) \in \mathbb{R}^{m+1}$ tashqi maydon.

Chekli $D \subset V$ soha uchun uning to‘ldiruvchisi $D^c = V \setminus D$ da berilgan chegaraviy konfiguratsiya φ_{D^c} bilan shartli Hamiltonian quyidagicha ifodalanadi

$$H(\sigma_D \mid \varphi_{D^c}) = -J \sum_{\substack{<x,y>, \\ x,y \in D}} |\sigma(x) - \sigma(y)| - J \sum_{\substack{<x,y>, \\ x \in D, y \in D^c}} |\sigma(x) - \varphi(y)| - J \sum_{x \in D} \alpha_{\sigma(x), x}.$$

$B - \Omega$ ning silindrik qism to’plamlaridan hosil bo‘lgan σ-algebra bo‘lsin. (Ω, B) dagi μ ehtimollik o‘lchovi Dobrushin-Lendford-Ryuel tenglamasini qanotlantirsa, unga Gibbs o‘lchovi deyiladi, ya’ni, barcha chekli $D \subset V$ qism to‘plamlar va $\sigma_D \in \Omega_D$ uchun quyidagi tenglik bajarilsa

$$\mu\left(\{\omega \in \Omega : \omega|_D = \sigma_D\}\right) = \int_\Omega \mu(d\varphi) \nu_\varphi^D(\sigma_D),$$

bunda ν_φ^D shartli ehtimollik o‘lchovi:

$$\nu_\varphi^D(\sigma_D) = \frac{1}{Z_{D,\varphi}} \exp\left(-\beta H(\sigma_D \mid \varphi_{D^c})\right).$$

Shuningdek, $\beta = \frac{1}{T}$, $T > 0$ - harorat va $Z_{D,\varphi}$ chegaraviy sharti φ bo‘lgan, D dagi normallovchi kattalik.

$x\in V\mapsto h_x=(h_{0,x},\ldots,h_{m,x})\in\mathbb{R}^{m+1}$ uchun biz (chekli o‘lchamli) Gibbs taqsimotini

$$\mu_n(\sigma_n)=Z_n^{-1}\exp\left\{-\beta H(\sigma_n)+\sum_{x\in W_n}h_{\sigma(x),x}\right\} \tag{2}$$

formula bilan aniqlaymiz, bunda Z_n normallovchi kattalik.

Agar barcha $\sigma_{n-1}\in\Omega_{V_{n-1}}$ lar uchun $\sum\limits_{\substack{\omega_n\in\Omega_{V_n}:\\ \omega_n|_{V_{n-1}}\equiv\sigma_{n-1}}}\mu_n(\omega_n)=\mu_{n-1}(\sigma_{n-1})$ tenglik o‘rinli bo‘lsa, (2) ehtimollik taqsimotlari muvofiq deyiladi. Bu shart ostida, Ω da shunday yagona μ o‘lchov mavjud bo‘ladiki, barcha n va $\sigma_n\in\Omega_{V_n}$ lar uchun $\mu(\{\sigma|_{V_n}=\sigma_n\})=\mu_n(\sigma_n)$ o’rinli. Bunday o‘lchov H Hamiltonian va $x\mapsto h_x$ funksiyaga mos splitting Gibbs o‘lchovi (SGO‘) deyiladi.

Quyidagi belgilashni kiritaylik $\alpha=\{\alpha_x,x\in V\}$.

1-ta’rif. Agar (β,α) juftlik uchun kamida ikkita turli Gibbs o‘lchovlari qurish mumkin bo‘lsa, (β,α) da faza almashishi mavjud deyiladi.

Faraz qilaylik, G_k barpo etuvchilari $a_1,a_2,\ldots,a_{k+1}$ bo‘lgan $k+1$ ta ikkinchi tartibli siklik gruppalarning ozod ko‘paytmasi bo‘lsin.

N.N. G‘anixo‘jayevning ishlaridan quyidagi natija ma’lum

1-tasdiq. Γ^k Keli daraxtining V uchlar to‘plami bilan G_k gruppa orasida o‘zaro bir qiymatli moslik mavjud.

2-ta’rif. K G_k ning qism gruppasi bo‘lsin. Agar barcha $x\in G_k$ va $y\in K$ uchun $h_{yx}=h_x$ bajarilsa, $h=\{h_x:x\in G_k\}$ funksiyalar jamlanmasi K-davriy deyiladi. G_k-davriy h funksiya translyation-invariant deyiladi.

3-ta’rif. Agar SGO‘i h ning K-davriy jamlanmasiga mos bo‘lsa, unga K-davriy deyiladi.

1-teorema. (2) kabi aniqlangan $\mu_n(\sigma_n),n=1,2,\ldots$ ehtimollik taqsimotlari muvofiq bo‘lishi uchun har bir $x\in V\setminus\{x^0\}$ uchun quyidagi vektor tenglama

$$\tilde{h}_x=\tilde{\alpha}_x+\sum_{y\in S(x)}F(\tilde{h}_y;\theta), \tag{3}$$

o‘rinli bo‘lishi zarur va yetarli, bunda

$$\theta=\exp(\beta J),\ \tilde{h}_x=(\tilde{h}_{0,x},...,\tilde{h}_{m-1,x}),\ \tilde{\alpha}_x=(\tilde{\alpha}_{0,x},...,\tilde{\alpha}_{m-1,x}),$$
$$\tilde{h}_{i,x}=h_{i,x}-h_{m,x}+\beta(\alpha_{i,x}-\alpha_{m,x}),$$
$$\tilde{\alpha}_{i,x}=\beta(\alpha_{i,x}-\alpha_{m,x}),\quad i=0,...,m-1,$$

$u=(u_0,...,u_{m-1})\in\mathbb{R}^m$ uchun $F(u,\theta)=(F_0(u,\theta),\ldots,F_{m-1}(u,\theta))$ akslantirish

$$F_i(u,\theta) := \ln \frac{\sum_{j=0}^{m-1} \theta^{|i-j|} \exp(u_j) + \theta^{m-i}}{\sum_{j=0}^{m-1} \theta^{m-j} \exp(u_j) + 1}, \quad i = 0,...,m-1$$

formulalar bilan aniqlanadi.

1-izoh. 1-teoremaga ko‘ra, Gibbs o‘lchovlarini tavsiflash masalasi (3) tenglamalar sistemasining yechimlarini tavsiflashga keltirilishini anglatadi.

(3) tenglamalar sistemasi $m = 2$ uchun quyidagicha bo‘ladi

$$\begin{cases} \tilde{h}_{0,x} = \tilde{\alpha}_{0,x} + \sum_{y \in S(x)} \ln \dfrac{\exp(\tilde{h}_{0,y}) + \theta \exp(\tilde{h}_{1,y}) + \theta^2}{\theta^2 \exp(\tilde{h}_{0,y}) + \theta \exp(\tilde{\tilde{h}}_{1,y}) + 1}, \\ \tilde{h}_{1,x} = \tilde{\alpha}_{1,x} + \sum_{y \in S(x)} \ln \dfrac{\theta \exp(\tilde{h}_{0,y}) + \exp(\tilde{h}_{1,y}) + \theta}{\theta^2 \exp(\tilde{h}_{0,y}) + \theta \exp(\tilde{h}_{1,y}) + 1}. \end{cases} \tag{4}$$

Biz (4) tenglamalar sistemasining translyatsion-invariant (TI) yechimlarini o‘rganamiz, ya’ni, $\tilde{h}_x = \tilde{h} \in \mathbb{R}^2$, $\tilde{\alpha}_x = \tilde{\alpha} \in \mathbb{R}^2, \forall x \in V$ deb faraz qilamiz.

$$\theta_c = \frac{k-1}{\sqrt{k^2 + 6k + 1}}, \ \lambda_1^* = \frac{2\theta^{k+1}}{c_2(\theta)} \text{ va } \lambda_2^* = \frac{2\theta^{k+1}}{c_1(\theta)}$$

belgilashlarni kiritaylik.

2-teorema. $k \geq 2$ bo‘lsin. Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun quyidagi tasdiqlar o‘rinli

- Agar $\theta \geq \theta_c$ yoki $\theta < \theta_c$ va $\lambda \notin \left[\lambda_1^*, \lambda_2^*\right]$ bo‘lsa, kamida 1 ta TISGO‘ i mavjud;
- Agar $\theta < \theta_c$ va $\lambda \in \left\{\lambda_1^*, \lambda_2^*\right\}$ bo‘lsa, kamida 2 ta TISGO‘ lari mavjud;
- Agar $\theta < \theta_c$ va $\lambda \in \left(\lambda_1^*, \lambda_2^*\right)$ bo‘lsa, kamida 3 ta TISGO‘ lari mavjud.

2-izoh. 1-ta’rifga ko‘ra, 2-teoremaning ikkinchi va uchinchi tasdiqlari fazalar almashishi mavjud bo‘lishini ta’minlaydi.

Quyidagi belgilashlarni kiritaylik

$$\lambda_1(\theta) = \frac{-71\theta^4 + 38\theta^2 + 1 - \sqrt{(1-\theta)(1+\theta)(1-17\theta^2)^3}}{16\theta}, \ \theta_1 = \sqrt{\frac{1}{17}},$$

$$\lambda_2(\theta) = \frac{-71\theta^4 + 38\theta^2 + 1 + \sqrt{(1-\theta)(1+\theta)(1-17\theta^2)^3}}{16\theta}, \ \theta_2 = \sqrt{\frac{1}{3}},$$

$$\lambda_3(\theta) = \frac{\theta^4 + 2\theta^2 + 1}{4\theta}, \ \lambda_4(\theta) = 4\theta\left(1 - 3\theta^2\right), \ \theta_3 = \sqrt{\frac{1}{7}}, \ \theta_4 = \frac{1}{\sqrt{19}}.$$

3-teorema. Ikkinchi tartibli Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun quyidagi tasdiqlar o‘rinli:

1. Agar $\theta \in (0,\theta_4)$ va $\lambda \in (\lambda_4(\theta),\lambda_2(\theta))$ bajarilsa, aniq yettita TISGO‘lari mavjud bo‘ladi;

2. Agar $\theta \in (0,\theta_4)$ va $\lambda = \lambda_2(\theta)$ bajarilsa, aniq oltita TISGO‘lari mavjud bo‘ladi;

3. Agar $\theta \in (0,\theta_4]$ va $\lambda \in (\lambda_2(\theta),\lambda_3(\theta))$ yoki $\theta \in (\theta_4,\theta_3)$ va $\lambda \in (\lambda_4(\theta),\lambda_3(\theta))$ bajarilsa, aniq beshta TISGO‘lari mavjud bo‘ladi;

4. Agar $\theta \in (0,\theta_4)$ va $\lambda \in (\lambda_1(\theta),\lambda_4(\theta)]$ yoki $\theta \in [\theta_4,\theta_1)$ va $\lambda \in (\lambda_1(\theta),\lambda_2(\theta))$ bajarilsa, aniq beshta TISGO‘lari mavjud bo‘ladi;

5. Agar $\theta \in (0,\theta_1)$ va $\lambda = \lambda_1(\theta)$ yoki $\theta \in [\theta_4,\theta_1)$ va $\lambda = \lambda_2(\theta)$ bajarilsa, aniq to‘rtta TISGO‘lari mavjud bo’ladi;

6. Agar $\theta \in (0,\theta_1)$ va $\lambda \in (0,\lambda_1(\theta))$ yoki $\theta \in (0,\theta_3)$ va $\lambda = \lambda_3(\theta)$ yoki $\theta \in (\theta_4,\theta_1)$ va $\lambda \in (\lambda_2(\theta),\lambda_4(\theta)]$ yoki $\theta \in [\theta_2,\theta_1)$ va $\lambda \in (0,\lambda_4(\theta))$ yoki $\theta \in [\theta_1,\theta_3)$ va $\lambda = \lambda_4(\theta)$ bajarilsa, aniq uchta TISGO‘lari mavjud bo‘ladi;

7. Boshqa hollarda yagona TISGO‘i mavjud bo‘ladi.

3-izoh. 3-teoremada fazalar almashishi sodir bo‘ladigan barcha sohalar topilgan.

Ikkinchi bob **“Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun davriy Gibbs o‘lchovlari”** deb nomlanadi va Keli daraxtida translyatsion-invariant va $G_k^{(2)}$-davriy tashqi maydonli uch holatli SOS modelining davriy Gibbs o‘lchovlarini o‘rganishga bag‘ishlanadi.

Biz uch holatli TI tashqi maydon ta’siridagi SOS modelining davriy Gibbs o‘lchovlarini tatqiq qilamiz. Buning uchun, (4) funksional tenglamaning davriy yechimlarini tashqi maydon TI bo‘lgan holda o‘rganamiz.

Ma’lumki, $G_k^{(2)}$ gruppa G_k ning uzunligi juft bo‘lgan barcha so‘zlaridan tashkil topgan qism gruppasidir.

4-teorema. K G_k ning chekli indeksli normal qism gruppasi bo‘lsin. U holda TI tashqi maydonli uch holatli SOS modeli uchun har bir K-davriy SGO‘ yoki TI, yoki $G_k^{(2)}$-davriy bo‘ladi.

$I = \{z = (z_0, z_1, t_0, t_1) \in R^4 : \lambda_0 = 1; z_0 = t_0 = 1\}$ belgilashni kiritaylik.

5-teorema. TI tashqi maydonli uch holatli SOS modeli uchun I invariant to‘plamda quyidagi tasdiqlar o‘rinli

- $\theta = 1$ uchun barcha $G_k^{(2)}$-davriy SGO‘lari TISGO‘lari bilan ustma-ust tushadi;
- $k \le 5$ uchun barcha $G_k^{(2)}$-davriy SGO‘lari TISGO‘lari bilan ustma-ust tushadi;
- $k > 5,\ \theta < 1$ uchun barcha $G_k^{(2)}$-davriy SGO‘lari TISGO‘lari bilan ustma-ust tushadi;
- $k > 5,\ \theta > 1$ uchun

1) Agar $\dfrac{k\zeta(\theta^2-1)(x^*)^{k-1}}{(\theta(x^*)^k+\theta^2+1)^2} > 1$ shart bajarilsa, u holda kamida uchta $G_k^{(2)}$-

davriy SGO‘lari mavjud bo‘ladi;

2) Agar $\frac{k\zeta(\theta^2-1)(x^*)^{k-1}}{(\theta(x^*)^k+\theta^2+1)^2}\le 1$ shart bajarilsa, u holda kamida bitta $G_k^{(2)}$-davriy SGO‘ mavjud bo’ladi, bunda x^* quyidagi tenglamaning yechimidir

$$x=\zeta\frac{x^k+2\theta}{\theta x^k+\theta^2+1}.$$

Endi Keli daraxtida $G_k^{(2)}$-davriy tashqi maydonli uch holatli SOS modeli uchun $G_k^{(2)}$-davriy Gibbs o‘lchovlarini o‘rganamiz. Buning uchun, (4) tenglamalar sistemasini $G_k^{(2)}$-davriy tashqi maydon ta’sirida tatqiq qilamiz.

$\tilde{h}=\{\tilde{h}_x : x\in G_k\}$ va tashqi maydonning $G_k^{(2)}$-davriy to’plami, mos ravishda, quyidagi ko‘rinishga ega:

$$\tilde{h}_x=\begin{cases}\tilde{h}_0, \text{ agar } x\in G_k^{(2)},\\ \tilde{h}_1, \text{ agar } x\in G_k\setminus G_k^{(2)},\end{cases}\quad \tilde{\alpha}_x=\begin{cases}\tilde{\alpha}_0, \text{ agar } x\in G_k^{(2)},\\ \tilde{\alpha}_1, \text{ agar } x\in G_k\setminus G_k^{(2)}.\end{cases}$$

$G_k^{(2)}$-davriy tashqi maydon ta’sirida (4) tenglamalar sistemasining $G_k^{(2)}$-davriy yechimlarini topish masalasini $I=\{z=(z_0,z_1,t_0,t_1)\in\mathbb{R}^4 : \lambda_0=1; \mu_0=1; z_0=t_0=1\}$ invariant to‘plam ustida $\psi(z)=\lambda_1 f\left(\mu_1 f\left(z\right)\right)$ funksiyaning qo‘zg‘almas nuqtalarini topish masalasiga keltirish mumkin, bunda

$$f\left(z\right)=\left(\frac{z+2\theta}{\theta z+\theta^2+1}\right)^k.$$

z^* qiymat $z=\psi\left(z\right)$ tenglaming yechimi bo‘lsin.

6-teorema. $k\ge 2$ tartibli Keli daraxtida $G_k^{(2)}$-davriy tashqi maydonli uch holatli SOS modeli uchun J to‘plamda agar $\psi'(z^*)>1$ shart bajarilsa, u holda kamida uchta SGO‘lari mavjud bo‘ladi, ya’ni faza almashishi sodir bo‘ladi.

Uchinchi bob **“Keli daraxtida uch holatli SOS modeli uchun chegaraviy shartlar”** deb nomlanib, chegaraviy shartlarga nisbatan limit Gibbs o‘lchovlari TISGO‘lari bilan mos keladigan chegaraviy konfiguratsiyalar to‘plamini olishga bag‘ishlangan.

$\Gamma^k=(V,L)$ tartibi $k\ge 2$ bo‘lgan Keli daraxti bo‘lsin. Agar x^0 va t uchlarni bog‘lovchi yo‘l s $(s\ne t)$ orqali o‘tsa, biz $s\to t$ kabi yozamiz. Agar $s\to t$ va s,t uchlar yaqin qo‘shni bo’lsa, t uch s uchning to‘g‘ri avlodi deyiladi va $s\to_1 t$ belgilashidan foydalanamiz.

A to‘plam V ning chekli qism to‘plami va $\omega\in\Omega$, $\sigma\in\Omega_A$ bo’lsin. Berilgan ichki konfiguratsiya σ va chegaraviy konfiguratsiya ω uchun A dagi SOS ta’sir energiyasi

$$H_A^{\omega}(\sigma)=-J\sum_{t,s\in A:\langle t,s\rangle}|\sigma(t)-\sigma(s)|-J\sum_{t\in A,s\in\partial A:\langle t,s\rangle}|\sigma(t)-\omega(s)|$$

kabi aniqlanadi, bunda $J \in \mathbb{R}$ ta'sirlashish o'zgarmasi. Ehtimollik o'lchovi

$$P_A^{\omega}(\sigma) = [Z_A^{\omega}]^{-1} \exp[-H_A^{\omega}(\sigma)], \ \sigma \in \Omega_A$$

kabi aniqlanadi, bunda $Z_A^{\omega} = \sum_{\tilde{\sigma} \in \Omega_A} \exp[-H_A^{\omega}(\tilde{\sigma})]$.

4-ta'rif. Agar ta'yinlangan J va mos $\omega \in \Omega$ uchun shunday chekli qism to'plamlarning o'suvchi $\{V_n\}$ ketma-ketligi mavjud bo'lib, $n \to \infty$ da $V_n \nearrow V$ va $P^{\omega} = \text{w-}\lim_{n\to\infty} P_{V_n}^{\omega}$ (o'lchovlarning kuchsiz yaqinlashishi) mavjud bo'lsa, u holda P^{ω} – ω chegaraviy shartga ega limit Gibbs o'lchovi deyiladi.

Faraz qilaylik, $c^l(\omega) = \sum_{s: t \to_1 s} |l - \omega(s)|$, $l = 0, 1, \ldots, m$ miqdor $t \in V \setminus \{x^0\}$ uchlarga bog'liq bo'lmasin.

$m = k = 2$ bo'lsin. Quyidagi tenglamani qaraylik

$$h_1 = 2\ln \frac{\exp(h_1) + 2\theta}{\theta^2 + 2\exp(h_1) + 1}. \tag{5}$$

1-lemma. Shunday $\theta_c \approx 0.1414$ mavjudki, quyidagi tasdiqlar o'rinli

- Agar $\theta < \theta_c$ bo'lsa, u holda (5) tenglama uchta $\alpha_3(\theta) < \alpha_2(\theta) < \alpha_1(\theta)$ yechimlarga ega bo'ladi;
- Agar $\theta = \theta_c$ bo'lsa, u holda (5) tenglama 2 ta $\alpha_3(\theta) = \alpha_2(\theta) < \alpha_1(\theta)$ yechimlarga ega bo'ladi;
- Agar $\theta > \theta_c$ bo'lsa, u holda (5) tenglama yagona $\alpha_1(\theta)$ yechimga ega bo'ladi.

Quyidagi to'plamlarni belgilaylik

$$\mathbb{D} = \{\omega \in \Omega : c^0(\omega) = c^2(\omega)\}, \ \mathbb{D}^+ = \{\omega \in \mathbb{D} : J(c^1(\omega) - c^2(\omega)) > \alpha_2\},$$

$$\mathbb{D}^0 = \{\omega \in \mathbb{D} : J(c^1(\omega) - c^2(\omega)) = \alpha_2\}, \ \mathbb{D}^- = \{\omega \in \mathbb{D} : J(c^1(\omega) - c^2(\omega)) < \alpha_2\}.$$

7-teorema. (i) Agar $\theta > \theta_c$ va $\omega \in \mathbb{D}$ bo'lsa, u holda $P^{\omega} = \mu_1(\theta)$.

(ii) Agar $\theta = \theta_c$ bo'lsa, u holda $P^{\omega} = \begin{cases} \mu_2(\theta), \text{ agar } \omega \in \mathbb{D}^0 \cup \mathbb{D}^-, \\ \mu_1(\theta), \text{ agar } \omega \in \mathbb{D}^+. \end{cases}$

(iii) Agar $\theta < \theta_c$ bo'lsa, u holda $P^{\omega} = \begin{cases} \mu_3(\theta), \text{ agar } \omega \in \mathbb{D}^-, \\ \mu_2(\theta), \text{ agar } \omega \in \mathbb{D}^0, \\ \mu_1(\theta), \text{ agar } \omega \in \mathbb{D}^+. \end{cases}$

μ_1 TISGO'i uchun chegaraviy konfiguratsiyalar quraylik. Agar $\theta \le \theta_c$ bo'lsa, biz $\omega \in \mathbb{D}^+$ kabi olishimiz kerak, ya'ni, $J(c^1(\omega) - c^2(\omega)) > \alpha_2$, $c^0(\omega) = c^2(\omega)$. U holda, chegaraviy konfiguratsiya sifatida $c^0(\omega) = c^2(\omega) = 2$ va $c^1(\omega) = 1$ shartlarni qanoatlantiradigan har qanday ω konfiguratsiyani olishimiz mumkin.

XULOSA

Dissertatsiya tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant, davriy splitting Gibbs o‘lchovlarini o‘rganishga, shuningdek, ikkinchi tartibli Keli daraxtda modelning ayrim TISGO‘lari uchun chegaraviy shartlarni qurishga bag‘ishlangan.

Tadqiqotda quyidagi asosiy natijalarga erishildi:

1. Ikkinchi tartibli Keli daraxtida tashqi maydonli uch holatli SOS modeli uchun translyatsion-invariant splitting Gibbs o‘lchovlari to‘plami to‘la tavsiflandi, ixtiyoriy tartibli Keli daraxtida bunday o‘lchovlarning yagona bo‘lmaslik shartlari topildi;

2. Ixtiyoriy tartibli Keli daraxtida translyatsion-invariant tashqi maydonli uch holatli SOS modeli uchun davriy Gibbs o‘lchovlari to‘la tavsiflandi hamda translyatsion-invariant bo‘lmagan, ikki davriy Gibbs o‘lchovining mavjudligi ko‘rsatildi;

3. Ixtiyoriy tartibli Keli daraxtida ikki davriy tashqi maydonli uch holatli SOS modeli uchun faza almashish mavjudligi ko‘rsatildi;

4. Limit Gibbs o‘lchovlari translyatsion-invariant splitting Gibbs o‘lchovlari bilan mos keladigan chegaraviy konfiguratsiyalar to‘plami tasniflandi.

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CHIRCHIK STATE PEDAGOGICAL UNIVERSITY

KARSHIBOEV OBID SHERKUL UGLI

BOUNDARY CONFIGURATIONS FOR GIBBS MEASURES OF SOME MODELS DEFINED ON LATTICES

01.01.01 – Mathematical analysis

ABSTRACT OF DISSERTATION OF THE DOCTOR OF PHILOSOPHY (PhD) ON PHYSICAL AND MATHEMATICAL SCIENCES

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O'ZBEKISTON RESPUBLIKASI FANLAR AKADEMIYASI V.I. ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI

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INTRODUCTION (abstract of PhD thesis)

Actuality and demand of the theme of dissertation. A number of scientific and practical tasks dedicated to the study of the thermodynamic properties of physical and biological systems, conducted on a global scale, are often led to the investigation of models of statistical mechanics. Gibbs measures are the main tool for studying the theory of phase transitions of statistical mechanics models. It is also important to apply Gibbs measures in solving problems in various fields of science and technology, such as chemistry, service theory, and materials science. The problem of fully describing the set of all Gibbs measures for classical models of statistical mechanics has not been fully solved for any model to date. Therefore, enriching the set of Gibbs measures for such models, including finding the set of boundary configurations for Gibbs measures of some models defined on the lattices, is one of the urgent tasks.

Nowadays, one of the urgent tasks is to demonstrate the existence of translation-invariant and periodic Gibbs measures corresponding to Hamiltonians with external fields for lattice systems and to fully describe the set of such Gibbs measures. Determining the non-uniqueness of translation-invariant and periodic measures in the set of Gibbs measures for a given physical model is of great importance in the theory of phase transitions of statistical physics. In this regard, finding all translation-invariant and periodic Gibbs distributions for a given Hamiltonian, classifying the set of boundary configurations where the limit Gibbs measures correspond to the translation-invariant splitting Gibbs measures, is one of the priority scientific studies.

In our country, attention is being paid to areas of scientific and practical application of fundamental sciences, which are one of the main pillars of development. In particular, in recent years, significant results have been achieved in determining the existence of translation-invariant, periodic, and non-periodic Gibbs measures for classical models of statistical mechanics on the Cayley tree, as well as in enriching the set of such measures. Conducting scientific research at the level of international standards in the priority areas of “Algebra and its applications, differential equations and its applications, nonlinear systems, dynamical systems, mathematical modeling, stochastic analysis, functional analysis, medical and biological informatics, computational mathematics” are defined as the main tasks and areas of activity of mathematical science[1]. To enforce the resolution, it is important to develop the theory of phase transitions for classical models of statistical mechanics with the aim of using scientific results in related fields of science.

The subject and object of the research of this dissertation are in line with tasks identified in the Decrees of the President of the Republic of Uzbekistan UP-4947 of February 7, 2017 “On the strategy of action for the further development of the

[1] Decree of the President of the Republic of Uzbekistan dated July 9, 2019 № PQ-4387 “On state support for the further development of mathematics education and subjects, as well as measures to fundamentally improve the activities of the Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan”.

Republic of Uzbekistan”, PP-60 of January 28, 2022, “On the Development Strategy of New Uzbekistan for 2022-2026”, UP-2789 dated February 17, 2017 “On measures to further improvement of the activities of the Academy of Sciences, organization, management and financing of research activities”, PP-4387 from July 9, 2019 “On measures to further development of mathematical education and science, and also root improvement of the activity of the Uzbekistan Academy of Sciences V.I.Romanovsky Institute of Mathematics”, UP-4708 dated May 7, 2020 “Quality of education in the field of mathematics”, PP-60 dated January 28, 2022, “On the Development Strategy of New Uzbekistan for 2022-2026” and this dissertation research serves to a certain extent in the implementation of the tasks specified in other normative-law documents related to above activities.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of Republic of Uzbekistan IV, “Mathematics, Mechanics and Computer Science”.

The degree of scrutiny of the problem. The concept of Gibbs distribution, which is important for systems that are in thermal equilibrium with their environment, was introduced by American scientist J.W. Gibbs. The basic concepts of Gibbs measures were studied in detail by R.L. Dobrushin, O. Lanford, and D. Ruelle. Modern achievements in the theory of Gibbs measures are widely covered in the research of R. Baxter, H.O. Georgii, V.A. Malyshev, R.A. Minlos, K. Preston, D. Ruelle, Ya.G. Sinai, G. Gallavotti, F. Bonetto.

In classical statistical mechanics models, Gibbs measures have been extensively studied by E.I. Dinaburg, Yu.M. Suhov, A.E. Mazel, L. Coquille, C. Kuelske, A. Le Ny, D. Gandolfo, N.N. Ganikhodjaev, U.A. Rozikov, F.M. Mukhamedov, M.M. Rahmatullaev, R.M. Khakimov, G.I. Botirov, O.N. Hakimov, N.M. Khatamov, Sh.A. Shoyusupov, F.H. Haydarov, M.A. Rasulova, A.M. Tukhtabaev, M.T. Makhammadaliev, B.U. Abraev, and many other researchers.

In the work of A.E. Mazel and Yu.M. Suhov, the solid-on-solid (SOS) model on the $\mathbb{Z}^d$ lattice was considered. This work proved the existence of a critical temperature that determines the structure of thermodynamic phases at low temperatures through the dominant ground states of the SOS model. Subsequently, E.I. Dinaburg and A.E. Mazel studied the SOS model with an external field on this lattice and found conditions under which phase transitions occur.

In the work of U.A. Rozikov and Yu.M. Suhov, the SOS model on the Cayley tree was considered, in which the critical temperature for the three-state model was found, and at lower temperatures, it was proven that there are at least three translation-invariant Gibbs splitting measures for this model. Furthermore, this work provides a complete description of periodic Gibbs measures. U.A. Rozikov studied the finite-state SOS model on the Cayley tree of order two using the contour argument (Pirogov-Sinai theory). It has been proven that at sufficiently low temperatures, Gibbs measures exist for the considered model, and their number is at least equal to the number of spin values. Later, G.I. Botirov and U.A. Rozikov generalized these results for the Cayley tree of arbitrary order.

C. Kuelske and U.A. Rozikov considered the three-state SOS model on the Cayley tree of order two. They classified all translation-invariant splitting Gibbs measures of the model under consideration. In addition, they check whether these measures are extreme or non-extreme whenever they exist. M.M. Rahmatullaev and B.U. Abraev studied the three-state SOS model on the Cayley tree of arbitrary order and constructed some non-translational-invariant Gibbs measures using the known translation-invariant Gibbs measures.

Y. Higuchi developed a method of constructing limiting Gibbs measures for the Ising model on the Cayley tree by changing the boundary conditions. This method provided an accurate way to construct different Gibbs measures corresponding to different equilibrium states of the system. D. Gandolfo, M.M. Rahmatullaev and U.A. Rozikov constructed boundary conditions for the translation-invariant splitting Gibbs measures of the finite state Potts model on the Cayley tree of order two.

It should be noted that despite numerous scientific works, a complete classification of all limiting Gibbs measures for any model on the Cayley tree has not yet been obtained.

Connection of the theme of the dissertation with the research works of scientific research Institute, where the dissertation is carried out. Dissertation research was carried out within the “Fundamental research” branch of the research plan of Chirchik state pedagogical university.

The aim of research work contains studying the translation-invariant, periodic Gibbs measures for three-state solid-on-solid (SOS) model with an external field on the Cayley tree, obtaining the boundary configurations to study the limiting Gibbs measures, and studying phase transitions.

Research problems:

A complete description of the set of translation-invariant splitting Gibbs measures for the three-state SOS model with an external field on a Cayley tree of order two, finding the conditions for the non-uniqueness of such measures on an arbitrary Cayley tree;

To give a complete description of the periodic Gibbs measures for the three-state SOS model with a translational-invariant external field on an arbitrary Cayley tree and to show the existence of a non-translational-invariant, two-periodic Gibbs measures;

To demonstrate the existence of a phase transition for the three-state SOS model with the two-periodic external field;

To classify a set of boundary configurations which the limiting Gibbs measures coincide with the translation-invariant splitting Gibbs measures.

The research object: the three-state SOS model with an external field on the Cayley tree.

The research subject: Theory of groups and graphs, Gibbs measures theory, algebra and number theory, theory of non-linear discrete dynamical systems, non-linear Markov processes.

Research methods: In the research the methods of number theory, functional analysis, combinatorics, group theory, measure theory, theory of linear algebra and dynamical systems are used.

The scientific novelty of the research consists of the followings:

The set of translation-invariant splitting Gibbs measures for the three-state SOS model with an external field on the Cayley tree of order two is completely described, and conditions for the non-uniqueness of such measures on the Cayley tree of arbitrary order are found;

Periodic Gibbs measures for the three-state SOS model with a translation-invariant external field on the Cayley tree of arbitrary order are completely described and the existence of a non-translation-invariant two-periodic Gibbs measure is shown;

The existence of a phase transition for a three-state SOS model with the two-periodic external field on a Cayley tree of arbitrary order has been shown;

A set of boundary configurations has been classified, in which limit Gibbs measures correspond to translation-invariant splitting Gibbs measures.

Practical results of the research consist of the followings:

A complete description of the set of all positive roots of the higher-order equation was used to determine the number of translation-invariant Gibbs splitting measures for the three-state SOS model of statistical mechanics with an external field. To prove the existence of periodic Gibbs measures for the three-state SOS model with an external field, methods were used that determine the nature of the fixed points of the function.

The reliability of the results of the study. The results have been obtained by using the methods of mathematical analysis, functional analysis, Markov theory of random fields, probability theory, number theory, discrete dynamical systems and measure theory. The obtained results are mathematically strongly proved.

Scientific and practical significance of research results.

The scientific significance of the research results is explained by the fact that the theory of Gibbs measures is the main tool in the study of problems of statistical mechanics in the development of science and technology.

The practical significance of the research results is explained by the fact that the changes in the state of physical systems have been studied and allowed to solve some problems in the theory of communication networks and neural networks.

Implementation of the research results. The scientific results obtained during the research of the dissertation are implemented in the following research projects:

translation-invariant, periodic Gibbs measures for a three-state solid-on-solid (SOS) model with an external field on the Cayley tree have been implemented on in the foreign grant project number G0003247 “Chaotic and mixing *p*-adic dynamical systems associated with renormalized groups of lattice models” is used to investigation of boundary configurations for the SOS and other related models on trees (United Arab Emirates University Bulletin dated 26 September 2024, UAE). Application of this scientific result made it possible to develop the new methodologies for investigating the boundary configurations for the SOS and other related models on trees.

The methodology for a complete description of the set of periodic Gibbs measures for the solid-on-solid (SOS) model with three states and a translation-invariant external field on the Cayley tree of arbitrary order, as well as the classification of the set of boundary configurations whose limit Gibbs measures coincide with the translation-invariant splitting Gibbs measures was used in the fundamental project 374874-2022 “Phase transition problems and critical phenomena. Mathematical aspects of their equations, fast transitions and asymptotics” in studying the presence of a phase transition in physical systems and for studying the dynamic properties of this system (Osh State University Certificate No. 1429 dated 05.11.2024, Kyrgyzstan). The application of this scientific result made it possible to study the asymptotic properties and periodic trajectories of dynamic systems with discrete time.

Approbation of the research results. The main results of the research have been discussed in 3 international and 4 national scientific conferences.

Publications of the research results. On the topic of the dissertation 5 research papers have been published in the scientific journals, all of them are included in the list of journals proposed by the Higher Attestation Commission of the Republic of Uzbekistan for defending the PhD thesis, including 4 in foreign and 1 in Republic journals.

The structure and volume of the dissertation. The dissertation consists of an introduction, three chapters, conclusion and bibliography. The volume of the thesis is 98 pages.

In the introduction besides the motivation of research theme and correspondence to the priority research areas of science and technology of the Republic, we present the degree of scrutiny of the problem, formulate our goals and objectives, identify the object and subject of study, and state scientific novelty and practical results of the research. Moreover, we reduce the theoretical and practical importance of the obtained results, and give information on the implementation of the research results, the published works and the structure of dissertation.

The first chapter of the thesis, titled “**Translation invariant splitting Gibbs measures for the three-state SOS model with an external field on the Cayley tree**”, is devoted present the obtained results on translation-invariant splitting Gibbs measures for the three-state SOS model with the translation-invariant external field on the Cayley tree of order higher than or equal to two.

The Cayley tree $\Gamma^k=(V,L)$ of order $k\geq 1$ with the root x^0 is an infinite tree, i.e., a cycles-free, connected and undirected graph, such that exactly $k+1$ edges originate from each vertex. Here V is the set of vertices and L is the set of edges. The vertices x and y are called nearest neighbors if there exists a edge $\langle x,y\rangle\in L$ connecting them. The sequence of different edges $\langle x,x_1\rangle,\dots,\langle x_{d-1},y\rangle$ is called the path from the vertex x to the vertex y. A path can be used to determine the metric on a tree, i.e., the number of edges on a path from the vertex x to the vertex y is called the distance between these edges and is denoted by $d(x,y)$.

We consider the following sets $V_n=\{x\in V: d(x,x^0)\le n\}$, $W_n=\{x\in V: d(x,x^0)=n\}$. For $x\in W_n$ we define the set $S(x)=\{y\in W_{n+1}: d(y,x)=1\}$. The set $S(x)$ is called the set of direct successors of x.

For $A\subseteq V$ a spin configuration σ_A on A is defined as a function $x\in A\to\sigma_A(x)\in\Phi=\{0,1,...,m\}$. The set of all configurations coincide with $\Omega_A=\Phi^A$. For the sake of simplicity, we denote $\Omega=\Omega_V$ and $\sigma=\sigma_V$.

The SOS model with an external field is defined by the formal Hamiltonian

$$H(\sigma)=-J\sum_{\langle x,y\rangle\in L}|\sigma(x)-\sigma(y)|-\sum_{x\in V}\alpha_{\sigma(x),x}, \tag{1}$$

where $J\in\mathbb{R}$ is the coupling constant and $\alpha_x=(\alpha_{0,x},\alpha_{1,x},...,\alpha_{m,x})\in\mathbb{R}^{m+1}$ is the external field.

For a finite domain $D\subset V$ with boundary condition φ_{D^c} given on its complement $D^c=V\setminus D$, the conditional Hamiltonian is

$$H\left(\sigma_D\,|\,\varphi_{D^c}\right)=-J\sum_{\substack{<x,y>,\\ x,y\in D}}\left|\sigma(x)-\sigma(y)\right|-J\sum_{\substack{<x,y>,\\ x\in D,y\in D^c}}\left|\sigma(x)-\varphi(y)\right|-J\sum_{x\in D}\alpha_{\sigma(x),x}.$$

Let B be σ-algebra generated by cylinder subsets of Ω. A probability measure μ on (Ω,B) is called a Gibbs measure if it satisfies the Dobrushin-Lanford-Ruelle equation: for all finite $D\subset V$ and $\sigma_D\in\Omega_D$:

$$\mu\left(\{\omega\in\Omega:\omega|_D=\sigma_D\}\right)=\int_\Omega\mu(d\varphi)\nu_\varphi^D(\sigma_D),$$

where ν_φ^D is the conditional probability:

$$\nu_\varphi^D(\sigma_D)=\frac{1}{Z_{D,\varphi}}\exp\left(-\beta H\left(\sigma_D\,|\,\varphi_{D^c}\right)\right).$$

Here $\beta=\frac{1}{T}$, $T>0$ is temperature and $Z_{D,\varphi}$ stands for the partition function in D, with the boundary condition φ:

$$Z_{D,\varphi}=\sum_{\tilde\sigma\in\Omega_D}\exp\left(-\beta H\left(\tilde\sigma_D\,|\,\varphi_{D^c}\right)\right).$$

For $x\in V\mapsto h_x=(h_{0,x},\dots,h_{m,x})\in\mathbb{R}^{m+1}$ we define the (finite-dimensional) Gibbs distributions by the formula

$$\mu_n(\sigma_n) = Z_n^{-1}\exp\{-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\} \tag{2}$$

where Z_n is the corresponding partition function.

The probability distributions (2) are compatible if for all $\sigma_{n-1}\in\Omega_{V_{n-1}}$ one has $\sum_{\substack{\omega_n\in\Omega_{V_n}:\\ \omega_n|_{V_{n-1}}\equiv\sigma_{n-1}}} \mu_n(\omega_n) = \mu_{n-1}(\sigma_{n-1})$. Under this condition, there exists a unique measure μ on Ω such that $\forall n$ and $\sigma_n\in\Omega_{V_n}$, $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n)$. Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian H and the function $x\mapsto h_x$.

We introduce the following notation $\alpha = \{\alpha_x, x\in V\}$.

Definition 1. If at least two distinct Gibbs measures can be constructed for a pair (β,α), we say that there is a (first-order) phase transition at (β,α).

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2,\dots,a_{k+1}$, respectively.

It is known from the works of N.N. Ganikhodjaev that

Proposition 1. There exists a one-to-one correspondence between the set of vertices V of the Cayley tree Γ^k and the group G_k.

Definition 2. Let K be a subgroup of G_k. We say that a collection of functions $h=\{h_x : x\in G_k\}$ is K-periodic if $h_{yx} = h_x$ for all $x\in G_k$ and $y\in K$. A G_k-periodic function h is called translation-invariant.

Definition 3. A SGM is called K-periodic if it corresponds to a K-periodic collection *h*.

Theorem 1. The probability distributions $\mu_n(\sigma_n), n=1,2,\dots$ defined in (2) are compatible if and only if for any $x\in V\setminus\{x^0\}$ the following vector equation holds

$$\tilde{h}_x = \tilde{\alpha}_x + \sum_{y\in S(x)} F(\tilde{h}_y;\theta), \tag{3}$$

here

$$\theta = \exp(\beta J),\ \tilde{h}_x = (\tilde{h}_{0,x},\dots,\tilde{h}_{m-1,x}),\ \tilde{\alpha}_x = (\tilde{\alpha}_{0,x},\dots,\tilde{\alpha}_{m-1,x}),$$
$$\tilde{h}_{i,x} = h_{i,x} - h_{m,x} + \beta(\alpha_{i,x} - \alpha_{m,x}),$$
$$\tilde{\alpha}_{i,x} = \beta(\alpha_{i,x} - \alpha_{m,x}),\quad i = 0,\dots,m-1,$$

the map $F(u,\theta) = (F_0(u,\theta),\dots,F_{m-1}(u,\theta))$ is defined for $u = (u_0,\dots,u_{m-1})\in\mathbb{R}^m$ and $\theta>0$ by the formulas

$$F_i(u,\theta) := \ln \frac{\sum_{j=0}^{m-1} \theta^{|i-j|} \exp(u_j) + \theta^{m-i}}{\sum_{j=0}^{m-1} \theta^{m-j} \exp(u_j) + 1}, \quad i = 0,\ldots,m-1.$$

Remark 1. From Theorem 1, it follows that the problem of describing the Gibbs measures is reduced to the descriptions of the solutions of the equation (3).

For $m = 2$, from (3) it follows that

$$\begin{cases} \tilde{h}_{0,x} = \tilde{\alpha}_{0,x} + \sum_{y \in S(x)} \ln \frac{\exp(\tilde{h}_{0,y}) + \theta \exp(\tilde{h}_{1,y}) + \theta^2}{\theta^2 \exp(\tilde{h}_{0,y}) + \theta \exp(\tilde{\tilde{h}}_{1,y}) + 1}, \\ \tilde{h}_{1,x} = \tilde{\alpha}_{1,x} + \sum_{y \in S(x)} \ln \frac{\theta \exp(\tilde{h}_{0,y}) + \exp(\tilde{h}_{1,y}) + \theta}{\theta^2 \exp(\tilde{h}_{0,y}) + \theta \exp(\tilde{h}_{1,y}) + 1}. \end{cases} \tag{4}$$

We study the translation-invariant (TI) solutions of (4), i.e., we assume $h_x = h \in \mathbb{R}^2,\ \tilde{\alpha}_x = \tilde{\alpha} \in \mathbb{R}^2, \forall x \in V$. Denote

$$\theta_c = \frac{k-1}{\sqrt{k^2+6k+1}},\ \lambda_1^* = \frac{2\theta^{k+1}}{c_2(\theta)} \text{ and } \lambda_2^* = \frac{2\theta^{k+1}}{c_1(\theta)}.$$

Theorem 2. Let $k \geq 2$. For the three-state SOS model with an external field on the Cayley tree the following assertions hold

- There is at least 1 TISGM if $\theta \geq \theta_c$ or $\theta < \theta_c$ and $\lambda \notin [\lambda_1^*, \lambda_2^*]$,
- There are at least 2 TISGMs if $\theta < \theta_c$ and $\lambda \in \{\lambda_1^*, \lambda_2^*\}$,
- There is at least 3 TISGMs if $\theta < \theta_c$ and $\lambda \in (\lambda_1^*, \lambda_2^*)$.

Remark 2. By Definition 1, the second and third assertions of Theorem 2 imply the existence of a phase transition.

Introduce the following notations

$$\lambda_1(\theta) = \frac{-71\theta^4 + 38\theta^2 + 1 - \sqrt{(1-\theta)(1+\theta)(1-17\theta^2)^3}}{16\theta},\ \theta_1 = \sqrt{\frac{1}{17}},$$

$$\lambda_2(\theta) = \frac{-71\theta^4 + 38\theta^2 + 1 + \sqrt{(1-\theta)(1+\theta)(1-17\theta^2)^3}}{16\theta},\ \theta_2 = \sqrt{\frac{1}{3}},$$

$$\lambda_3(\theta) = \frac{\theta^4 + 2\theta^2 + 1}{4\theta},\ \lambda_4(\theta) = 4\theta\left(1-3\theta^2\right),\ \theta_3 = \sqrt{\frac{1}{7}},\ \theta_4 = \frac{1}{\sqrt{19}}.$$

Theorem 3. For the three-state SOS model with the external field on the Cayley tree of order two the following assertions hold:

1. There are exactly seven TISGMs if $\theta \in (0, \theta_4)$ and $\lambda \in (\lambda_4(\theta), \lambda_2(\theta))$.
2. There are exactly six TISGMs if $\theta \in (0, \theta_4)$ and $\lambda = \lambda_2(\theta)$.
3. There are exactly five TISGMs

(a) if $\theta \in (0,\theta_4]$ and $\lambda \in (\lambda_2(\theta),\lambda_3(\theta))$;

(b) if $\theta \in (\theta_4,\theta_3)$ and $\lambda \in (\lambda_4(\theta),\lambda_3(\theta))$.

4. There are exactly another five such measures

(a) if $\theta \in (0,\theta_4)$ and $\lambda \in (\lambda_1(\theta),\lambda_4(\theta)]$;

(b) if $\theta \in [\theta_4,\theta_1)$ and $\lambda \in (\lambda_1(\theta),\lambda_2(\theta))$.

5. There are exactly four such measures

(a) if $\theta \in (0,\theta_1)$ and $\lambda = \lambda_1(\theta)$;

(b) if $\theta \in [\theta_4,\theta_1)$ and $\lambda = \lambda_2(\theta)$.

6. There are exactly three such measures

(a) if $\theta \in (0,\theta_1)$ and $\lambda \in (0,\lambda_1(\theta))$;

(b) if $\theta \in (0,\theta_3)$ and $\lambda = \lambda_3(\theta)$;

(c) if $\theta \in (\theta_4,\theta_1)$ and $\lambda \in (\lambda_2(\theta),\lambda_4(\theta)]$;

(d) if $\theta \in [\theta_2,\theta_1)$ and $\lambda \in (0,\lambda_4(\theta))$;

(e) if $\theta \in [\theta_1,\theta_3)$ and $\lambda = \lambda_4(\theta)$.

7. Otherwise there exists a unique such measure.

Remark 3. In Theorem 3 we find all regions where the phase transition occurs.

The second Chapter, titled "**Periodic Gibbs measures for the three-state SOS model with an external field on the Cayley tree**", is devoted to the study of periodic Gibbs measures of three-state SOS model with the translation-invariant and "chess-board" external fields on the Cayley tree.

We investigate the periodic Gibbs measures of the three-state SOS model under the TI external field. To do this, we study periodic solutions of the functional equation (4) with TI external field.

It is known that $G_k^{(2)}$ is a subgroup of G_k consisting of all words of even length.

Theorem 4. Let K be a normal subgroup of finite index in G_k. Then each K-periodic SGM for three-state SOS model with TI external field is either TI or $G_k^{(2)}$-periodic.

Introduce the notation $J = \{z = (z_0, z_1, t_0, t_1) \in \mathbb{R}^4 : \lambda_0 = 1; z_0 = t_0 = 1\}$.

Theorem 5. For the three-state SOS model with TI external field on the invariant set J the following assertions hold

- For $\theta = 1$ all $G_k^{(2)}$-periodic SGMs coincide with TISGMs;
- For $k \le 5$ all $G_k^{(2)}$-periodic SGMs coincide with TISGMs;
- For $k > 5$, $\theta < 1$ all $G_k^{(2)}$-periodic SGMs coincide with TISGMs;
- For $k > 5$, $\theta > 1$

1) If the condition $\dfrac{k\zeta(\theta^2-1)(x^*)^{k-1}}{(\theta(x^*)^k+\theta^2+1)^2} > 1$ holds then there exist at least three $G_k^{(2)}$-periodic SGMs;

2) If the condition $\frac{k\zeta(\theta^2-1)(x^*)^{k-1}}{(\theta(x^*)^k+\theta^2+1)^2}\leq 1$ hold then there exists at least one $G_k^{(2)}$-periodic SGM, where x^* is a solution of the following equation

$$x=\zeta\frac{x^k+2\theta}{\theta x^k+\theta^2+1}.$$

Now, we study the periodic Gibbs measures for the three-state SOS model on a Cayley tree under a $G_k^{(2)}$-periodic external field. To do this, we consider the functional equation (4) with a $G_k^{(2)}$-periodic external field.

A $G_k^{(2)}$-periodic set of $\tilde{h}=\{\tilde{h}_x : x\in G_k\}$ and the external field have the following forms, respectively,

$$\tilde{h}_x=\begin{cases}\tilde{h}_0, \text{ if } x\in G_k^{(2)},\\ \tilde{h}_1, \text{ if } x\in G_k\setminus G_k^{(2)},\end{cases}\quad \tilde{\alpha}_x=\begin{cases}\tilde{\alpha}_0, \text{ if } x\in G_k^{(2)},\\ \tilde{\alpha}_1, \text{ if } x\in G_k\setminus G_k^{(2)}.\end{cases}$$

Finding the $G_k^{(2)}$-periodic solutions of the system of equation (4) under $G_k^{(2)}$-periodic external field on the invariant set $I=\{z=(z_0,z_1,t_0,t_1)\in\mathbb{R}^4 : \lambda_0=1;\mu_0=1;z_0=t_0=1\}$ can be reduced to finding the fixed points of the function $\psi(z)=\lambda_1 f\left(\mu_1 f\left(z\right)\right)$, where

$$f\left(z\right)=\left(\frac{z+2\theta}{\theta z+\theta^2+1}\right)^k.$$

Let z^* be a solution of the equation $z=\psi\left(z\right)$.

Theorem 6. For the three-state SOS model with periodic external field on the Cayley tree of order $k\geq 2$ on the set I if the condition $\psi'(z^*)>1$ is satisfied, then there exists at least three SGMs, i.e., the phase transition occurs.

The third Chapter, titled "**Boundary conditions for the three-state SOS model on the Cayley tree**", is devoted to get a set of boundary conditions such that the limiting GMs with respect to the boundary conditions coincide with the TISGMs.

Let $\Gamma^k=(V,L)$ be the Cayley tree of order $k\geq 2$. We write $s\to t$, if $t\neq s$ and the path connecting x^0 and t passes through s. If $s\to t$ and s,t are neighbors, then t is called a direct successor of s and we write $s\to_1 t$.

Let A be a finite subset of $V, \omega\in\Omega$ and $\sigma\in\Omega_A$. The SOS interaction energy on A given the inner configuration σ and boundary condition ω is defined by

$$H_A^{\omega}(\sigma)=-J\sum_{t,s\in A:\langle t,s\rangle}|\sigma(t)-\sigma(s)|-J\sum_{t\in A,s\in\partial A:\langle t,s\rangle}|\sigma(t)-\omega(s)|,$$

where $J\in\mathbb{R}$. The probability measure is defined by

$$P_A^{\omega}(\sigma)=[Z_A^{\omega}]^{-1}\exp[-H_A^{\omega}(\sigma)],\sigma\in\Omega_A,$$

where $Z_A^\omega = \sum_{\tilde\sigma\in\Omega_A} \exp[-H_A^\omega(\tilde\sigma)]$.

Definition 4. For fixed J, if there is an increasing sequence of finite subsets $\{V_n\}$ such that $V_n \nearrow V$ as $n\to\infty$ and $P^\omega = \text{w-}\lim_{n\to\infty} P_{V_n}^\omega$ (the weak convergence of measures) exists for suitable fixed $\omega\in\Omega$, then P^ω is called a limiting Gibbs measure with boundary condition ω for J.

Assume that the quantity $c^l(\omega) = \sum_{s:t\to_1 s} |l-\omega(s)|$, $l = 0,1,\dots,m$ be independent of $t\in V\setminus\{x^0\}$.

Let $m = k = 2$. Consider the following equation

$$h_1 = 2\ln\frac{\exp(h_1)+2\theta}{\theta^2+2\exp(h_1)+1}. \tag{5}$$

Lemma 1. There exists $\theta_c \approx 0.1414$ such that

- If $\theta<\theta_c$ then the equation (5) has three solutions $\alpha_3(\theta)<\alpha_2(\theta)<\alpha_1(\theta)$;
- If $\theta=\theta_c$ then the equation (5) has two solutions $\alpha_3(\theta)=\alpha_2(\theta)<\alpha_1(\theta)$;
- If $\theta>\theta_c$ then the equation (5) has a unique solution $\alpha_1(\theta)$.

We denote the following sets:

$$\mathbb{D} = \{\omega\in\Omega : c^0(\omega) = c^2(\omega)\},\ \mathbb{D}^+ = \{\omega\in\mathbb{D} : J(c^1(\omega)-c^2(\omega)) > \alpha_2\},$$
$$\mathbb{D}^0 = \{\omega\in\mathbb{D} : J(c^1(\omega)-c^2(\omega)) = \alpha_2\},$$
$$\mathbb{D}^- = \{\omega\in\mathbb{D} : J(c^1(\omega)-c^2(\omega)) < \alpha_2\}.$$

Theorem 7. (i) If $\theta>\theta_c$ and $\omega\in\mathbb{D}$ then $P^\omega = \mu_1(\theta)$.

(ii) If $\theta=\theta_c$ then

$$P^\omega = \begin{cases} \mu_2(\theta), \text{if } \omega\in\mathbb{D}^0\cup\mathbb{D}^-, \\ \mu_1(\theta), \text{if } \omega\in\mathbb{D}^+. \end{cases}$$

(iii) If $\theta<\theta_c$ then

$$P^\omega = \begin{cases} \mu_3(\theta), \text{if } \omega\in\mathbb{D}^-, \\ \mu_2(\theta), \text{if } \omega\in\mathbb{D}^0, \\ \mu_1(\theta), \text{if } \omega\in\mathbb{D}^+. \end{cases}$$

Let us construct several boundary configurations for the TISGM μ_1. If $\theta\le\theta_c$, one has to take $\omega\in\mathbb{D}^+$, i.e., $J(c^1(\omega)-c^2(\omega))>\alpha_2$, $c^0(\omega)=c^2(\omega)$, where $J=\ln\theta$. In this case, one can take a configuration ω such that $c^0(\omega)=c^2(\omega)=2$ and $c^1(\omega)=1$.

CONCLUSION

The dissertation is devoted to the study of translation-invariant, periodic splitting Gibbs measures for the three-state SOS model with an external field, as well as constructing the boundary conditions for some TISGMs of the model on the Cayley tree of order two.

The main results of the research are as follows:

1. The set of translation-invariant Gibbs splitting measures for the three-state SOS model with an external field on the Cayley tree of order two is completely described, and conditions for the non-uniqueness of such measures on the Cayley tree of arbitrary order are found;

2. Periodic Gibbs measures for the three-state SOS model with a translation-invariant external field on the Cayley tree of arbitrary order are completely described and the existence of a non-translation-invariant two-periodic Gibbs measures is shown;

3. The existence of a phase transition for a three-state SOS model with the two-periodic external field on a Cayley tree of arbitrary order has been shown;

4. A set of boundary configurations has been classified, in which limit Gibbs measures correspond to translation-invariant splitting Gibbs measures;

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ЧИРЧИКСКИЙ ГОСУДАРСТВЕННЫЙ ПЕДАГОГИЧЕСКИЙ УНИВЕРСИТЕТ

КАРШИБОЕВ ОБИД ШЕРКУЛ УГЛИ

ГРАНИЧНЫЕ КОНФИГУРАЦИИ ДЛЯ МЕР ГИББСА НЕКОТОРЫХ МОДЕЛЕЙ, ОПРЕДЕЛЕННЫХ НА РЕШЕТКАХ

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АВТОРЕФЕРАТ ДИССЕРТАЦИИ ДОКТОРА ФИЛОСОФИИ (PhD) ПО ФИЗИКО-МАТЕМАТИЧЕСКИМ НАУКАМ

ТАШКЕНТ – 2025

Тема диссертации доктора философии (Doctor of Philosophy) по физико-математическим наукам зарегистрирована в Высшей аттестационной комиссии при Министерстве Высшего образования, Науки и Инноваций Республики Узбекистан за № B2021.3.PhD/FM622.

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Автореферат диссертации на трёх языках (узбекский, английский, русский (резюме)) размещен на веб-странице Научного совета (http://kengash.mathinst.uz) и на Информационно-образовательном портале «ZiyoNet» (www.ziyonet.uz).

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С диссертацией можно ознакомиться в Информационно-ресурсном центре Института Математики имени В.И.Романовского (зарегистрирована за № 198). (Адрес: 100174, г. Ташкент, Алмазарский район, ул. Университетская, 9. Тел.: (+998 78) 207 91 40).

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(протокол рассылки № 2 от «24» января 2025 года).

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ВВЕДЕНИЕ (аннотация диссертации доктора философии (PhD))

Целью исследования являются изучение трансляционно-инвариантных, периодических мер Гиббса для модели solid-on-solid (SOS) с тремя состояниями и внешним полем на дереве Кэли, получение граничных конфигураций для изучения предельных мер Гиббса и изучение фазовых переходов.

Объектом исследования является модель SOS с тремя состояниями и внешним полем на дереве Кэли.

Научная новизна исследования состоит в следующем:

Полностью описан набор трансляционно-инвариантных расщепленных мер Гиббса для модели SOS с тремя состояниями и внешним полем на дереве Кэли второго порядка, а также найдены условия неединственности таких мер на дереве Кэли произвольного порядка;

Полностью описаны периодические меры Гиббса для модели SOS с тремя состояниями и трансляционно-инвариантным внешним полем на дереве Кэли произвольного порядка, а также показано существование нетрансляционно-инвариантной двух-периодической меры Гиббса;

Доказано существование фазового перехода для SOS модели с тремя состояниями и двух-периодическим внешним полем на дереве Кэли произвольного порядка;

Классифицирован набор граничных конфигураций, в которых предельные меры Гиббса соответствуют трансляционно-инвариантным расщепленным мерам Гиббса.

Внедрение результатов исследования. Научные результаты, полученные в ходе диссертационного исследования, внедрены в следующие научно-исследовательские проекты:

трансляционно-инвариантные и периодические меры Гиббса модели solid-on-solid (SOS) с тремя состояниями и внешним полем на дереве Кэли были реализованы в зарубежном грантовом проекте G0003247 “Хаотические и перемешивающие *p*-адические динамические системы, связанные с перенормированными группами решеточных моделей”, который используется для исследования граничных конфигураций для SOS и других родственных моделей на деревьях (Бюллетень Университета Объединенных Арабских Эмиратов от 26 сентября 2024 г., ОАЭ). Применение данного научного результата позволило разработать новые методы исследования граничных конфигураций для SOS и других смежных моделей на деревьях.

Методика для полного описания набора периодических мер Гиббса для модели «solid-on-solid» (SOS) с тремя состояниями и трансляционно-инвариантным внешним полем на дереве Кэли произвольного порядка, а также классификация набора граничных конфигураций, при которых предельные меры Гиббса совпадают с трансляционно-инвариантными расщепленными мерами Гиббса использовались в фундаментальном проекте 374874-2022 “Задачи фазовых переходов и критические явления. Математические аспекты их уравнений, быстрые переходы и асимптотика” при изучении наличия фазового перехода в физических системах и для изучения динамических свойств этой системы (справка Ошского государственного университета № 1429 от 05.11.2024, Кыргызстан). Применение этого научного результата дало возможность изучить асимптотические свойства и периодические траектории динамических систем с дискретным временем.

Структура и объем диссертации. Диссертация состоит из введения, трех глав, заключения и списка использованной литературы. Объем диссертации составляет 98 страниц.

E'lon qilingan ishlar ro'yxati
List of published works
Список опубликованных работ

1-bo'lim (part 1, часть 1)

1. Rahmatullaev M.M., Karshiboev O.Sh. Translation-invariant Gibbs measures for the SOS model with external field on a Cayley tree // Доклады Академии Наук Республики Узбекистан. (2021), № 3, pp. 3–6. (01.00.00; № 7).

2. Rahmatullaev M.M., Karshiboev O.Sh. Gibbs measures for the three-state SOS model with external field on a Cayley tree // Positivity. (2022), Vol. 26, № 74, pp. 1–15. (3. Scopus. IF=1.0).

3. Karshiboev O.Sh. Periodic Gibbs measures for the three-state SOS model on a Cayley tree with a translation-invariant external field // Theoretical and Mathematical Physics. (2022), Vol. 212, № 3, pp. 1276–1283. (3. Scopus. IF=0.8).

4. Rahmatullaev M.M., Karshiboev O.Sh. Phase transition for the SOS model under inhomogeneous external field on a Cayley tree // Phase Transitions. (2022), Vol. 95, № 12, pp. 901–907. (3. Scopus. IF=1.465).

5. Rahmatullaev M.M., Karshiboev O.Sh. The boundary condition problems for the three-state SOS model on the binary tree // Lobachevskii Journal of Mathematics. (2023), Vol. 44, pp. 2891–2897. (3. Scopus. IF=1.013).

2-bo'lim (part 2, часть 2)

6. Rahmatullaev M.M., Karshiboev O.Sh. Ikkinchi tartibli Keli daraxtida aniqlangan SOS modeli uchun kuchsiz davriy Gibbs o'lchovlari // "Globallashuv davrida matematika va amaliy matematikaning dolzarb masalalari" mavzusidagi Respublika ilmiy anjumani. Toshkent, 1-2 iyun, 2021-yil, b. 183-186.

7. Rahmatullaev M.M., Karshiboev O.Sh. Functional equations for the SOS model with external field on a Cayley tree // Республиканская научная конференция "Сарымсаковские чтения". Ташкент, 16–18 сентября, 2021 г., с. 259-261.

8. Rahmatullaev M.M., Karshiboev O.Sh., On Gibbs measures for the SOS model with non-translation-invariant external field on a Cayley tree // Abstracts of the Conference "Modern problems of applied mathematics and information technologies al-Khwarizmi 2021". Fergana, November 15-17, 2021, p. 94

9. Rahmatullaev M.M., Karshiboev O.Sh. Phase transition for three-state SOS model with an external field on a Cayley tree of order two // Abstracts of the conference "Theoretical foundations and applied problems of modern mathematics". Andijon, March 17–19, 2022, pp. 79-82.

10. Rahmatullaev M.M., Karshiboev O.Sh. Periodic Gibbs measures for three-state SOS model with periodic external field on a Cayley tree // Abstracts of the Conference "New theorems of young mathematicians". Namangan, May 13–14, 2022, pp. 121–122

11. Rahmatullaev M.M., Karshiboev O.Sh. Description of the translation-invariant splitting Gibbs measures for the three-state SOS model on the binary tree // Abstracts of the International Conference “Mathematical analysis and its applications in modern mathematical physics”. Samarkand, September 23-24, 2022, pp. 102-104

12. Rahmatullaev M.M., Karshiboev O.Sh. Phase transitions for the three-state SOS model with periodic external field // Abstracts of the International Conference “Improving the mechanisms of tax administration and international taxation in a digital economy”. Tashkent, November 4, 2022, pp. 168-168.

Avtoreferat “O‘zbekiston matematika jurnali” tahririyatida 2024-yil 25 -dekabr tahrirdan o‘tkazilib, o‘zbek, rus va ingliz tillaridagi matnlar o‘zaro muvofiqlashtirildi.

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